Next, a result of Euler:

**Def.** Let \( n \in \mathbb{Z} \). Then:

\[
\begin{align*}
\text{p}_{\text{odd}} (n) : &= \text{(\# of partitions of } n \text{ into odd parts)}; \\
\text{p}_{\text{dist}} (n) : &= \text{(\# of partitions of } n \text{ into distinct parts)}. \\
\end{align*}
\]

**Ex.:**

\[
\text{p}_{\text{odd}} (7) = |\{ (7), (3, 3, 1), (3, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1) \} | = 5;
\]

\[
\text{p}_{\text{dist}} (7) = |\{ (7), (6, 1), (5, 2), (4, 3), (4, 2, 1) \} | = 5.
\]

**Thm. 9.2 (Euler),** \( \text{p}_{\text{odd}} (n) = \text{p}_{\text{dist}} (n) \quad \forall n \in \mathbb{N} \).

**1st proof:**

\[
\sum_{n \geq 0} \text{p}_{\text{odd}} (n) x^n = \prod_{k \geq 1} \frac{1}{1-x^k} \quad (\text{analogous to Thm. 9.1});
\]

\[
\sum_{n \geq 0} \text{p}_{\text{dist}} (n) x^n = \prod_{k \geq 1} (1+x^k) \quad (\text{also analogous to Thm. 9.1})
\]

Thus, it remains to prove
\[
\prod_{k \geq 1; \ k \text{ odd}} \frac{1}{1-x^k} = \prod_{k \geq 1} (1+x^k).
\]

First, prove
\[
\prod_{k \geq 1} \frac{1}{1-x^k} = (1+x)(1+x^2)(1+x^4)(1+x^8) \ldots
\]
\[
= \prod_{i \geq 0} (1+x^{2^i})
\]

1st proof of (88):
\[\text{RHS} = \sum_{k} x^k. \quad \text{(\# of ways to write } k \text{ as a sum of distinct powers of 2)}\]
\[= 1 \quad \text{(by binary representation)}\]

2nd proof of (88):
\[
(1-x) \cdot (1+x)(1+x^2)(1+x^4)(1+x^8) \quad \ldots
\]
\[
= \frac{1-x^2}{1-x^2} \cdot \frac{1+x^2}{1+x^2} \cdot \frac{1+x^4}{1+x^4} \cdot \frac{1+x^8}{1+x^8} \quad \ldots
\]
\[
= \frac{1}{1-x^8} \cdot \frac{1}{1+x^8} \quad \ldots
\]
\[
= \ldots \ldots = 1.
\]

More rigorously: To prove that \((1-x) \cdot \text{RHS} = 1\), it suffices to show that \([x^n]((1-x) \cdot \text{RHS}) = 0 \forall \text{ positive } n\).

For any given positive \(n\), we need to only perform finitely many steps of the computation above until we get an expression which has \(x^n\) as the highest power of \(x\) that appears.

For any odd \(k \geq 1\), we can substitute \(x^k\) for \(x\) in \((88)\), thus we get
\[
\frac{1}{1-x^k} = \prod_{i=0}^\infty (1+x^{k \cdot 2^i}).
\]

Multiplying these over all odd \(k\), we get
\[
\prod_{\substack{k \geq 1, \\
k \text{ odd}}} \frac{1}{1-x^k} = \prod_{\substack{k \geq 1, \\
k \text{ odd}}} \prod_{i=0}^\infty (1+x^{k \cdot 2^i}) = \prod_{m \geq 1} (1+x^m)
\]

(since each \(m \geq 1\) can be represented uniquely as \(k \cdot 2^i\) with odd \(k \geq 1\) and arbitrary \(i \geq 0\))

\[
= \prod_{k \geq 1} (1+x^k), \text{ so (87) is proven.}
\]

2nd proof (sketch). Construct a bijection

A: \{partitions of \(n\) into odd parts\} \(\rightarrow\) \{partitions of \(n\) into distinct parts\},

which transforms a partition by repeatedly merging 2 equal parts until no more equal parts can be found.
Ex: \((5, 5, 3, 1, 1, 1) \mapsto (10, 3, 1, 1, 1) \mapsto (10, 3, 2, 1)\).

Ex: \((5, 3, 1, 1, 1, 2) \mapsto (5, 3, 2, 1, 2) \mapsto (5, 3, 2, 2) \mapsto (5, 4, 3)\).

Why this is well-defined: not obvious.

One way to prove this is using the diamond lemma.

Another way is by representing each partition of our set as \(k \cdot 2^i\) with odd \(k \geq 1\) and arbitrary \(i \geq 0\).

This lets us analyze \(A\) in terms of binary representation.

\(A\) is called the Glaisher bijection.

The inverse of \(A\) transforms a partition by repeatedly splitting even parts into two equal pieces.

Prop. 9.3. Let \(n \in \mathbb{N}\) and \(k \geq 0\). Then,

\[p_k(n) = (\# \text{ of partitions of } n \text{ whose largest part is } k).\]
Proof sketch. Picture proof: e.g., let $k = 4$ and $n = 14$.

Start with the partition $\lambda = (5, 4, 4, 1)$ of $n$ into $k$ parts.

Draw a table of $k$ left-aligned rows, where the length of each row equals the corresponding part of $\lambda$:

\[
\begin{align*}
5 \rightarrow & \quad \square \quad \square \quad \square \quad \square \quad \square \\
4 \rightarrow & \quad \square \quad \square \quad \square \\
4 \rightarrow & \quad \square \quad \square \\
1 \rightarrow & \quad \square
\end{align*}
\]

This table is called the Young diagram or Fenner's diagram of $\lambda$.

Now, flip the table around the \ diagonal:
The lengths of the rows of the resulting table again form a partition of $n$, whose largest part is $k$. (In our example, this is $(4, 3, 3, 3, 1)$.) This is a bijection (and is called conjugation).

9.2. Euler's theorem.

**Def.** For any $k \in \mathbb{Z}$, define $w_k \in \mathbb{N}$ by

$$w_k = \frac{(3k-1)k}{2}.$$

This is called a pentagonal number.

**Thm. 9.4.** (Euler's pentagonal number theorem),

$$\prod_{k=1}^{\infty} (1-x^k) = \sum_{k \geq 2} (-1)^k x^{w_k}.$$

Thus,
\[
\prod_{k=1}^{\infty} (1-x^k) = \cdots + x^6 - x^5 + x^7 - x^2 + 1 - x + x^5 - x^{12} + x^{22} - x^{35} \pm \cdots
\]

\[
= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \pm \cdots
\]

Cor. 9.5.
\[
p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) \pm \cdots
\]

\[
= \sum_{k \neq 0} (-1)^{k-1} p(n-\omega_k) \quad \forall n > 0.
\]

Proof of Cor. 9.5 using Thm. 9.4.

(90) \[
\left( \sum_{m \geq 0} p(m) x^m \right) \left( \sum_{k \geq 2} (-1)^{k-1} x^{\omega_k} \right) = 1,
\]

\[
= \prod_{k \geq 1} \frac{1}{1-x^k} \quad \text{(by Thm. 9.1)}
\]

\[
= \prod_{k \geq 1} (1-x^k) \quad \text{(by Thm. 9.4)}
\]

Now, \( \omega_n \) is the \( x^n \)-coefficient on the LHS.
\[ \sum_{\substack{m \geq 0; \\ k \in \mathbb{Z}; \\ m + \omega_k = n}} p(m) \cdot (-1)^k = \sum_{\substack{k \in \mathbb{Z}; \\ n - \omega_k \geq 0}} p(n - \omega_k) \cdot (-1)^k \]

\[ = \sum_{k \in \mathbb{Z}} p(n - \omega_k) \cdot (-1)^k \quad \text{(since } p(n - \omega_k) = 0 \text{ when } n - \omega_k < 0) \]

\[ = \sum_{k \in \mathbb{Z}} (-1)^k \cdot p(n - \omega_k) \]

\[ = (-1)^0 \cdot p(n - \omega_0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^k \cdot p(n - \omega_k) \]

\[ = p(n) + \sum_{k \neq 0} (-1)^k \cdot p(n - \omega_k) \cdot \]
But (for $n > 0$) the $x^n$-coefficient on the RHS is 0 of Eq. (50).

Comparing yields

$$p(n) + \sum_{k=0}^{\infty} (-1)^k \ p(n-k) = 0. \quad \square$$

Solve this for $p(n)$.

Rmk. A combinatorial proof of Cor. 9.5 appears in May 2 slides.

9.3. Jacobi’s triple product identity

We will prove a stronger result (than Thm. 9.4):

Thm. 9.6 (Jacobi’s triple product identity),

$$\prod_{n=0}^{\infty} \left( \left( 1 + q^{2n-1} z \right) \left( 1 + q^{2n-1} z^{-1} \right) \left( 1 - q^{2n} \right) \right) = \sum_{\ell \in \mathbb{Z}} \ell q^{\ell^2} z^\ell.$$

(The best way to make this rigorous is to formalize both...
sides as formal power series in $q$ over the ring of Laurent series in $z$.

Alternatively, for what we will actually use them for, it suffices to take $q = x^a$ and $z = k \cdot x^b$ for two integers $a \& b$ with $a \geq |b|$ and $k \in \mathbb{Q}$, you can check that all of the factors & addends in Thm. 9.6 become proper FPS in $x$ in this case.)

Proof of Thm. 9.4 using Thm. 9.6. Set $q = x^3$ and $z = -x$

in Thm. 9.6, you get

$$\prod_{n>0} \left( (1 - \frac{x}{(2n+1)^{3/2}}) \left(1 - x^{(2n+1)^3 - 1} \right) \left(1 - x^{(2n+1)^3} \right) \right) = \left( \frac{(-1)^n}{3^n n!} \right) \sum_{l \in \mathbb{Z}} \frac{(-1)^l}{3^{l^2} l!} \chi_l$$

But the LHS of this equality is simply

$$\prod_{k=1}^{\infty} \left( 1 - x^{2k} \right)$$
since each $2k$ (with $k > 1$) can be uniquely represented either as $(2n-1)3 + 1$ or as $(2n-1)3 - 1$ or as $(2n+1)3$
for $n > 0$. So the equality becomes

\[
\frac{1}{1-x^{2k}} = \sum_{k=1}^{\infty} (-1)^{k} \frac{x^{3k^2-k}}{x^{k+1}} = \sum_{k=1}^{\infty} (-1)^{k} x^{3k^2-k}
\]

(since $3k^2-k = \frac{2k^3 - k}{2}$

(\since 3k^2-k

\begin{equation}
= \sum_{k=1}^{\infty} (-1)^{k} x^{2\omega_k},
\end{equation}

Thus, substituting $x^{1/2}$ for $x$ in this equality (= using the fact that any two FPs $f$ & $g$ satisfying $fox^2 = g o x^2$ must be equal), we obtain

\[
\frac{1}{1-x^{k}} = \sum_{k=1}^{\infty} (-1)^{k} x^{\omega_k}.
\]

\qed
Proof of Thm. 9.6. This comes from Cameron's AC-notes, going back to Borchers.

A level means a number of the form \( p + \frac{1}{2} \) with \( p \in \mathbb{Z} \).

A state \( \mathcal{S} \) is a set of levels which contains:

1. all but finitely many negative levels, and
2. only finitely many positive levels.

\[
\begin{align*}
\text{Ex:} & \quad \text{negative levels} \quad \cdot \quad \text{positive levels} \quad \cdot \\
& \quad \circ \circ \circ \times \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ
\end{align*}
\]

Legend:
0 = a level contained in the state \( \mathcal{S} \) = "electrons";
X = not contained = "holes".

For any state \( \mathcal{S} \), we define:

1. the energy of \( \mathcal{S} \) to be \( \sum_{p > 0, p \in \mathcal{S}} p - \sum_{p < 0, p \in \mathcal{S}} p \in \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right\} \).
the particle number of $s$ to be

\[(\# \text{ of } p > 0 \text{ such that } p \not\in s) - (\# \text{ of } p < 0 \text{ such that } p \not\in s)\]

$\in \mathbb{Z}$.

[In the example above:

the energy is $\frac{1}{2} + \frac{3}{2} + \frac{7}{2} + \frac{13}{2} - \frac{3}{2} - \frac{7}{2} = 7$,

the particle number is $4 - 2 = 2$.]

We want to prove Thm. 9.6. \#Rewrite it by replacing $q$ by $q^{1/2}$:

\[\prod_{n=0}^{i} \left(1 + q^{n-1/2} \varepsilon \right) \left(1 + q^{n-1/2} \varepsilon^{-1} \right) \left(1 - q^{n} \right) = \sum_{\ell \in \mathbb{Z}} q^{\ell^{2}/2} \varepsilon^{\ell}.

Moving the $(1 - q^n)$'s to the RHS, we rewrite this as
\[
\prod_{n=0} \left( \left( 1 + q^{n^{2} - 1} \right) \left( 1 + q^{n^{2} - 1} - 1 \right) \right) = \left( \sum_{\ell \leq 2} q^{\ell^{2} / 2} + 1 \right) \left( \prod_{n=0} (1-q^n)^{-2} \right).
\]

**(93)**

Claim 1: Let me state \( \ell \),

Let \( \frac{1}{2} N := \{ \frac{0}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots \} \).

**Claim 2:** Let \( m \in \frac{1}{2} N \) and \( \ell \leq 2 \).

(2) The coefficient of \( q^m \) on the LHS of (93) is the \# of states with energy \( m \) & \( \ell \) particle number \( \ell \).

(6) Same for the RHS.

Once claim 2 is proven, (93) will follow, & thus Thm. 9.6 will be proven.
Proof of Claim 1:

(2) \[ \text{LHS} = \prod_{n>0} \left(1 + q^{n-1/2} z^n \right) \prod_{n>0} \left(1 + q^{n-1/2} z^{-n} \right) \]

\[ = \prod_{m \text{ is a level}} \left(1 + q^{m/2} z \right) \prod_{p \text{ is a positive level}} \left(1 + q^p z \right) \prod_{p \text{ is a negative level}} \left(1 + q^{-p} z^{-1} \right) \]

If we expand this product, we get a sum over all states \( s \), the addend is \( q \) \( s \), energy(s) \( q \) \( s \), particle number(s) \( q \) \( s \).

Claim 1 (2) follows.
(b) \[ \text{RHS} = \left( \sum_{l \geq 2} q^{l^{2}/2} z^l \right) \left( \prod_{n>0} \left( 1 - q^n \right)^{-1} \right) \]

\[ = \sum_{n>0} \prod_{\lambda \text{ is a partition}} p(n) \# q^n \]

\[ = \sum_{\lambda \text{ is a partition}} q^{12} \]

(where \( |\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_k \)

for any partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \))

\[ = \left( \sum_{l \geq 2} q^{l^{2}/2} z^l \right) \sum_{\lambda \text{ is a partition}} q^{12} \]

\[ = \sum_{l \geq 2} \sum_{\lambda \text{ is a partition}} q^{l^{2}/2 + |\lambda|} z^l \]

Thus, in order to prove Claim 1 (b), we need to find a bijection

\[ \Phi: \{ \text{partitions } \lambda \text{ with } \ell^2/2 + |\lambda| = m \} \rightarrow \{ \text{states with energy } m \text{ & particle number } \ell \} \]

for fixed \( m, \ell \).

We define the state \( G_{\ell} \) ("the \( \ell \)-ground state") by

\[ G_{\ell} = \{ \text{all levels} < \ell \} \]

The energy of \( G_{\ell} \) is \( \frac{1}{2} + \frac{3}{2} + \ldots + \frac{2\ell-1}{2} = \frac{\ell^2}{2} \), and its particle number is \( \ell \).

If \( S \) is a state, and if \( p \leq S \), and if \( q \leq S \), then \( q \leq S \) is a positive integer such that \( p+q \leq S \), we let

\[ \text{jump}_{p,q}(S) = (S \setminus \{p+3\}) \cup \{p+q\} \]
we say that \( \text{jump}_{p,q}(s) \) is obtained from \( s \) by letting the electron at level \( p \) jump \( q \) steps (to the right). Note that \( \text{jump}_{p,q}(s) \) has the same particle number as \( s \), whereas its energy is \( q \) higher than that of \( s \).

So a jumping particle raises the energy but keeps the particle number unchanged.

For any partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), we define the state \( E_{l,\lambda} \) (called an "excited state") to be the state

\[
\text{jump}_{l-\lambda_1+\frac{1}{2}, \lambda_1} \left( \ldots \left( \text{jump}_{l-\lambda_k+\frac{1}{2}, \lambda_k} \left( \text{jump}_{l} \right) \right) \ldots \right)
\]

\[
= \{ \text{all levels } < l-k \} \cup \{ l-i+\frac{1}{2} + \lambda_i \mid i \in [k] \}
\]

This state \( E_{l,\lambda} \) has energy \( \frac{l^2}{2} + |\lambda| \) and particle number \( l \). Furthermore, every state with particle number \( l \) can be
written as $E_{n, \lambda}$ for a unique partition $\lambda$.

$\Rightarrow$ We get a bijection

$\text{sets partitions } \lambda \text{ with } \ell^2/2 + |\lambda| = m^2$

$\Rightarrow$ state with energy $m$ & particle number $\ell^2$

$\lambda \mapsto E_{n, \lambda}$.

This completes the proof of Claim 1(b). $\blacksquare$

So Thm. 9.6 is proven.

Reading recommendation: