1 Exercise 1

1.1 Problem

Let \( n \) be a positive integer.

An \( n \)-tuple \((i_1, i_2, \ldots, i_n)\) \(\in\{0, 1, 2, 3\}^n\) is said to be \textit{even} if the sum \(i_1 + i_2 + \cdots + i_n\) is even. (For example, the 4-tuple \((2, 3, 1, 2)\) is even, whereas \((1, 2, 3, 1)\) is not.)

Compute the number of all even \(n\)-tuples \((i_1, i_2, \ldots, i_n)\) \(\in\{0, 1, 2, 3\}^n\).

(Here and in all future exercises, all answers need to be proven.)

[\textbf{Hint}: Compare with Exercise 3 on \underline{Homework set #0}]

1.2 Solution

[...]

2 EXERCISE 2

2.1 PROBLEM

Let \( n \in \mathbb{N} \).

An \( n \)-tuple \((i_1, i_2, \ldots, i_n) \in \{0, 1, 2\}^n \) is said to be \textit{even} if the sum \( i_1 + i_2 + \cdots + i_n \) is even. (For example, the 4-tuple \((2, 1, 1, 2)\) is even, whereas \((1, 2, 2, 2)\) is not.)

Let \( e_n \) be the number of all even \( n \)-tuples \((i_1, i_2, \ldots, i_n) \in \{0, 1, 2\}^n \).

Prove that \( e_n = \frac{3^n + 1}{2} \).

[\textbf{Hint:} Induction on \( n \).]

2.2 SOLUTION

[...]

3 EXERCISE 3

3.1 PROBLEM

For any real number \( x \) and any \( k \in \mathbb{N} \), we define the lower factorial \( x^k \) as in Exercise 2 of Homework set #0. (Thus, \( x^k = x(x-1)(x-2)\cdots(x-k+1) = \prod_{i=0}^{k-1} (x-i) \). This boils down to \( x^0 = 1 \) when \( k = 0 \), since empty products are defined to be 1.)

Let \( k, a \) and \( b \) be three positive integers such that \( k \leq a \leq b \). Prove that

\[
(k-1) \sum_{i=a}^{b} \frac{1}{i^k} = \frac{1}{(a-1)^{k-1}} - \frac{1}{b^{k-1}}. \tag{1}
\]

3.2 REMARK

Remark 3.1. This is similar to Exercise 2 of Homework set #0, but here the lower factorials are in the denominators. The analogous fact from calculus is

\[
(k-1) \int_{a}^{b} \frac{1}{x^k} dx = \frac{1}{a^{k-1}} - \frac{1}{b^{k-1}}.
\]

3.3 SOLUTION

[...]

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4 Exercise 4

4.1 Problem

Definition 4.1. The Fibonacci sequence is the sequence \((f_0, f_1, f_2, \ldots)\) of integers which is defined recursively by \(f_0 = 0, f_1 = 1,\) and

\[
f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2.
\]

Here is a table of some of its first terms:

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_n)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
</tr>
</tbody>
</table>

Let \(n \in \mathbb{N}\). Recall some definitions from class:

Let \(R_{n,2}\) denote the set \([n] \times [2]\), which we regard as a rectangle of width \(n\) and height 2 (by identifying the squares with pairs of coordinates).

A *vertical domino* is a set of the form \(\{(i, j), (i, j + 1)\}\) for some \(i \in \mathbb{Z}\) and \(j \in \mathbb{Z}\).

A *horizontal domino* is a set of the form \(\{(i, j), (i + 1, j)\}\) for some \(i \in \mathbb{Z}\) and \(j \in \mathbb{Z}\).

A *domino tiling* of \(R_{n,2}\) means a set of disjoint dominos (i.e., vertical dominos and horizontal dominos) whose union is \(R_{n,2}\).

For example, there are 5 domino tilings of \(R_{4,2}\), namely

\[
\begin{align*}
& \{ \}, \\
& \{ \{1,1\}, \{2,2\}\}, \\
& \{ \{1,2\}, \{2,1\}\}, \\
& \{ \{1,3\}, \{2,3\}\}, \\
& \{ \{1,4\}, \{2,4\}\}\).
\end{align*}
\]

Written as a set of dominos, the second of these tilings is

\[
\{(1,1), (1,2)\}, \{(2,1), (2,2)\}, \{(3,1), (4,1)\}, \{(3,2), (4,2)\}\}
\]

We have seen in class (September 5) that the number of domino tilings of \(R_{n,2}\) is \(f_{n+1}\).

We have also counted “axisymmetric” domino tilings.

Let us now define a different kind of symmetry: A domino tiling \(S\) of \(R_{n,2}\) is said to be *centrosymmetric* if reflecting it across the center of the rectangle \(R_{n,2}\) leaves it unchanged. (Formally, if \(S\) is regarded as a set, it means that for every domino \(\{(i, j), (i', j')\}\) \(\in S\), its “opposite domino” \(\{(n + 1 - i, 3 - j), (n + 1 - i', 3 - j')\}\) is also in \(S\).) For example, among the 5 domino tilings of \(R_{4,2}\) listed above, exactly 3 are centrosymmetric (namely, the first, the fourth and the fifth).

Let \(s_n\) be the number of centrosymmetric domino tilings of \(R_{n,2}\).

(a) Prove that \(s_n = f_{(n+1)/2}\) if \(n\) is odd.

(b) Prove that \(s_n = f_{n/2+2}\) if \(n\) is even.

(Note that these are the same numbers as for axisymmetric domino tilings!)

[Hint: This is a bit of a trick problem.]
5 EXERCISE 5

5.1 PROBLEM
Let $n \in \mathbb{N}$. Let $S_{n,2}$ be the set
$$(n + 1] \times [2]) \setminus \{(1,2),(n + 1,1)\}.$$ For example, here is how $S_{6,2}$ looks like:

Find the number of domino tilings of $S_{n,2}$.

5.2 SOLUTION

6 EXERCISE 6

6.1 PROBLEM
Let $n \in \mathbb{N}$. If $S$ is a finite nonempty set of integers, then $\max S$ denotes the maximum of $S$ (that is, the largest element of $S$).

(a) Find the number of nonempty subsets $S$ of $[n]$ satisfying $\max S = |S|$.

(b) Find the number of nonempty subsets $S$ of $[n]$ satisfying $\max S = |S| + 1$.

[Hint: Exercise 7 from Spring 2018 Math 4707 Homework set #1 is similar to part (a), but uses the minimum instead of the maximum. Does this mean the answer should be similar?]
7 Exercise 7

7.1 Problem

For any nonnegative integers \(a\) and \(b\) and any real \(x\), prove that

\[
x^a x^b = \sum_{r=\max\{a,b\}}^{a+b} \frac{a!b!}{(r-a)!(r-b)!(a+b-r)!} x^r.
\] (4)

[**Hint:** Induction. First show the identities \(x^k = x^{k-1}(x - k + 1)\) and \(xx^k = x^{k+1} + kx^k\).]

7.2 Solution

[...]