1 Exercise 1

1.1 Problem
Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let $i$ and $j$ be two elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Let $Q$ be the set of all $k \in \{i+1, i+2, \ldots, j-1\}$ satisfying $\sigma(i) > \sigma(k) > \sigma(j)$. Prove that

$$\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - 2|Q| - 1.$$ 

1.2 Remark
This exercise implies that, in particular, $\ell(\sigma \circ t_{i,j}) < \ell(\sigma)$; this answers the question on page 213 of the notes from class (2018-10-22).

1.3 Solution

[...]


2 Exercise 2

2.1 Problem

Let \( n \in \mathbb{N} \) and \( \pi \in S_n \).

(a) Prove that

\[
\sum_{1 \leq i < j \leq n; \ \pi(i) > \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \ \pi(i) > \pi(j)} (i - j).
\]

(b) Prove that

\[
\sum_{1 \leq i < j \leq n; \ \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \ \pi(i) < \pi(j)} (j - i).
\]

[Hint: Exercise 5.18 in [Grinbe16] says something about sums of the form appearing in part (a). (See also Nathaniel Gorski’s solution of the same exercise in Spring 2018 Math 4707 homework set #4.) You may want to use the result or the ideas.]

2.2 Solution

[...]

3 Exercise 3

3.1 Problem

Let \( n \) be a positive integer. For each \( p \in \mathbb{Z} \), we let

\[
D_{n,p} = \{ \sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents} \}.
\]

(Recall that a descent of a permutation \( \sigma \in S_n \) denotes an element \( k \in [n-1] \) satisfying \( \sigma(k) > \sigma(k+1) \).)

Let \( p \in \mathbb{Z} \). Prove that \( |D_{n,p}| = |D_{n,n-1-p}| \).

3.2 Solution

[...]

4 Exercise 4

4.1 Problem

Let \( n \in \mathbb{N} \). Let \( S = \{ s_1 < s_2 < \cdots < s_k \} \) be a subset of \([n-1]\). Set \( s_0 = 0 \) and \( s_{k+1} = n \). For each \( i \in [k+1] \), set \( d_i = s_i - s_{i-1} \). (You might remember this construction from the definition of the map \( D \) in the solution to Exercise 1 on homework set #0.)
(a) Prove that

\[ \left| \{ \sigma \in S_n \mid \text{Des} \sigma \subseteq S \} \right| = \binom{n}{d_1, d_2, \ldots, d_{k+1}}. \]

(The term on the right hand side is a multinomial coefficient. The Des \( \sigma \) on the left hand side denotes the descent set of \( \sigma \), that is, the set of all descents of \( \sigma \).)

(b) Prove that

\[ \left| \{ \sigma \in S_n \mid \text{Des} \sigma = S \} \right| = \sum_{T \subseteq S} (-1)^{|S|-|T|} \left| \{ \sigma \in S_n \mid \text{Des} \sigma \subseteq T \} \right|. \]

4.2 Solution

[...]

5 Exercise 5

5.1 Problem

Let \( n \in \mathbb{N} \). We shall follow the convention that \( t_{i,i} \) denotes the identity permutation \( \text{id} \in S_n \) for each \( i \in [n] \).

Let \( \sigma \in S_n \).

It is known that there is a unique \( n \)-tuple \((i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]\) satisfying \( \sigma = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n} \). (See [Grinbe16, Exercise 5.9] for the proof of this fact, or – easier – do it on your own.) Consider this \( n \)-tuple. (It is sometimes called the transposition code of \( \sigma \).)

For each \( k \in \{0, 1, \ldots, n\} \), we define a permutation \( \sigma_k \in S_n \) by \( \sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} \).

Note that this permutation \( \sigma_k \) leaves each of the numbers \( k+1, k+2, \ldots, n \) unchanged (since all of \( i_1, i_2, \ldots, i_k \), as well as \( 1, 2, \ldots, k \), are \( \leq k \)).

For each \( k \in [n] \), let \( m_k = \sigma_k(k) \).

(a) Show that \( m_k \in [k] \) for all \( k \in [n] \).

(b) Show that \( \sigma_k(i_k) = k \) for all \( k \in [n] \).

(c) Show that \( \sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n} \).

(d) Let \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) be any \( 2n \) numbers. Prove that

\[ \sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} x_k y_{\sigma(k)} = \sum_{k=1}^{n} (x_k - x_{k'}) (y_{m_k} - y_k). \]

(e) Now assume that the numbers \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) are real and satisfy \( x_1 \geq x_2 \geq \cdots \geq x_n \) and \( y_1 \geq y_2 \geq \cdots \geq y_n \). Conclude that

\[ \sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\sigma(k)}. \]
5.2 Remark

Parts (a) and (c), combined, show that \((m_1, m_2, \ldots, m_n)\) is the transposition code of \(\sigma^{-1}\).

Part (e) of the exercise is known as the rearrangement inequality. The proof in this exercise is far from its easiest proof, but has the advantage of “manifest positivity” – i.e., it gives an explicit formula for the difference between the two sides as a sum of products of nonnegative numbers.

5.3 Solution

[...]

6 Exercise 6

6.1 Problem

Prove the following:

(a) If \(m \in \mathbb{N}\) and \(n \in \mathbb{N}\) are such that \(m < n\), then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0.
\]

(b) If \(n \in \mathbb{N}\) and \(r \in [n-1]\), then
\[
\sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0.
\]

6.2 Solution

[...]

7 Exercise 7

7.1 Problem

Let \(n \in \mathbb{N}\) and \(d \in \mathbb{N}\). An \(n\)-tuple \((x_1, x_2, \ldots, x_n) \in [d]^n\) is said to be all-even if each element of \([d]\) occurs an even number of times in this \(n\)-tuple (i.e., if for each \(k \in [d]\), the number of all \(i \in [n]\) satisfying \(x_i = k\) is even). For example, the 4-tuple \((1, 4, 4, 1)\) and the 6-tuples \((1, 3, 3, 5, 1, 5)\) and \((2, 4, 2, 4, 3, 3)\) are all-even, while the 4-tuples \((1, 2, 2, 4)\) and \((2, 4, 6, 4)\) are not.

Prove that the number of all all-even \(n\)-tuples \((x_1, x_2, \ldots, x_n) \in [d]^n\) is
\[
\frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d-2k)^n.
\]
[Hint: Compute the sum \[ \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n \] in two ways. One way is to split it according to the number of \( i \in [d] \) satisfying \( e_i = -1 \); this is a number \( k \in \{0, 1, \ldots, d\} \). Another way is by using the product rule:

\[
(e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}
\]

and then simplifying each sum \[ \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} \] using a form of destructive interference. This is not unlike the number of 1-even \( n \)-tuples, which we computed at the end of the 2018-10-10 class.]

7.2 SOLUTION

[...]

REFERENCES


The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2018-10-03.

Darij Grinberg, 00000000 dgrinber@umn.edu