1 Exercise 1

1.1 Problem
Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let $i$ and $j$ be two elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Let $Q$ be the set of all $k \in \{i+1, i+2, \ldots, j-1\}$ satisfying $\sigma(i) > \sigma(k) > \sigma(j)$. Prove that

$$\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - 2|Q| - 1.$$ 

1.2 Remark
This exercise implies that, in particular, $\ell(\sigma \circ t_{i,j}) < \ell(\sigma)$; this answers the question on page 213 of the notes from class (2018-10-22).

1.3 Solution
Jacob Elafandi gives a somewhat laborious but simple solution in [Elafan18].
I give a different solution in [Grinbe16, Exercise 5.20].
2 Exercise 2

2.1 Problem

Let \( n \in \mathbb{N} \) and \( \pi \in S_n \).

(a) Prove that
\[
\sum_{1 \leq i < j \leq n; \, \pi(i) > \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \, \pi(i) > \pi(j)} (i - j).
\]

(b) Prove that
\[
\sum_{1 \leq i < j \leq n; \, \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \, \pi(i) < \pi(j)} (j - i).
\]

2.2 Solution

We shall use the following fact:

Proposition 2.1. Let \( n \in \mathbb{N} \). Let \( \sigma \in S_n \). Let \( a_1, a_2, \ldots, a_n \) be any \( n \) numbers. (Here, “number” means “real number” or “complex number” or “rational number”, as you prefer; this makes no difference.) Prove that
\[
\sum_{1 \leq i < j \leq n; \, \sigma(i) > \sigma(j)} (a_j - a_i) = \sum_{i=1}^{n} a_i (i - \sigma(i)).
\]

[Here, the summation sign “\( \sum \)" means “\( \sum \)”; this is a sum over all inversions of \( \sigma \).]

Proposition 2.1 is [Grinbe16, Exercise 5.23]. For a different proof of it, see [Gorski18, Exercise 4].

Now, let us solve the exercise. We have \( \pi \in S_n \). In other words, \( \pi \) is a permutation of \([n]\). In other words, \( \pi \) is a bijection \([n] \to [n]\). Hence, we can substitute \( \pi(i) \) for \( i \) in the sum \( \sum_{i \in [n]} i^2 \). We thus obtain
\[
\sum_{i \in [n]} i^2 = \sum_{i \in [n]} (\pi(i))^2. \tag{1}
\]

(a) Proposition 2.1 (applied to \( \sigma = \pi \) and \( a_k = \pi(k) + k \)) yields
\[
\sum_{1 \leq i < j \leq n; \, \pi(i) > \pi(j)} ((\pi(j) + j) - (\pi(i) + i)) = \sum_{i=1}^{n} (\pi(i) + i) (i - \pi(i)) = \sum_{i \in [n]} (i^2 - (\pi(i))^2)
\]
for any two numbers \( x \) and \( y \)
\[
\sum_{i \in [n]} (i^2 - (\pi(i))^2) = \sum_{i \in [n]} i^2 - \sum_{i \in [n]} (\pi(i))^2 = 0
\]
Thus, for any $\pi$ of $(an equality between summation signs). Now, part (b) yields

$$0 = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} ((\pi(j) + j) - (\pi(i) + i)) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} ((\pi(j) - \pi(i)) - (i - j)) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (\pi(j) - \pi(i)) - \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j).$$

Adding $\sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j)$ to both sides of this equality, we obtain

$$\sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (\pi(j) - \pi(i)).$$

This solves part (a) of the exercise.

(b) Let $w_0$ denote the permutation in $S_n$ which sends each $k \in [n]$ to $n + 1 - k$. Define a permutation $\sigma \in S_n$ by $\sigma = w_0 \circ \pi$. Thus, each $k \in [n]$ satisfies

$$\sigma \circ (k) = (w_0 \circ \pi)(k) = w_0(\pi(k)) = n + 1 - \pi(k) \quad (2)$$

(by the definition of $w_0$).

For any $(i, j) \in [n]^2$, we have the following chain of logical equivalences:

\[
\begin{align*}
&\quad \sigma(i) > \sigma(j) \quad (by (2)) \\
&\quad \iff (n + 1 - \pi(i) > n + 1 - \pi(j)) \quad (by (2)) \\
&\quad \iff (\pi(i) < \pi(j)).
\end{align*}
\]

Thus, for any $(i, j) \in [n]^2$, the condition $(\sigma(i) > \sigma(j))$ is equivalent to $(\pi(i) < \pi(j))$. Hence, the summation sign “$\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)}$” can be rewritten as “$\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)}$”. In other words, we have

$$\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)} = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)}$$

(an equality between summation signs). Now, part (a) of the exercise (applied to $\sigma$ instead of $\pi$) yields

$$\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)} (\sigma(j) - \sigma(i)) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (i - j) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (-(j - i))$$

$$= \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j - i).$$
Comparing this with

\[
\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)} \left( \begin{array}{cc}
\sigma(j) & -\sigma(i) \\
= n+1-\pi(j) & = n+1-\pi(i)
\end{array} \right)
\]

\[
= \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} ((n+1-\pi(j)) - (n+1-\pi(i))) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (-\pi(j) - \pi(i))
\]

\[
= -\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (\pi(j) - \pi(i)),
\]

we obtain

\[
-\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = -\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j-i).
\]

Thus,

\[
\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j-i).
\]

This solves part (b) of the exercise.

### 3 Exercise 3

#### 3.1 Problem

Let \( n \) be a positive integer. For each \( p \in \mathbb{Z} \), we let

\[
D_{n,p} = \{ \sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents} \}.
\]

(Recall that a descent of a permutation \( \sigma \in S_n \) denotes an element \( k \in [n-1] \) satisfying \( \sigma(k) > \sigma(k+1) \).)

Let \( p \in \mathbb{Z} \). Prove that \( |D_{n,p}| = |D_{n,n-1-p}| \).

#### 3.2 Solution Sketch

We have \( n-1 \in \mathbb{N} \) (since \( n \) is a positive integer).

Recall that if \( \sigma \in S_n \) is a permutation, then \( \text{Des} \sigma \) denotes the set of all descents of \( \sigma \).

Let \( w_0 \) denote the permutation in \( S_n \) which sends each \( k \in [n] \) to \( n+1-k \).
Let $\pi \in S_n$. Thus, for each $k \in [n-1]$, we have the following chain of equivalences:

\[
(k \in \text{Des}(w_0 \circ \pi)) \iff (k \text{ is a descent of } w_0 \circ \pi) \iff \begin{pmatrix}
(w_0 \circ \pi)(k) > (w_0 \circ \pi)(k+1) \\
= w_0(\pi(k)) = n+1 - \pi(k) \\
(\text{by the definition of } w_0)
\end{pmatrix} \iff \begin{pmatrix}
(n+1 - \pi(k) > n+1 - \pi(k+1)) \\
(\pi(k) < \pi(k+1)) \iff (\pi(k) \leq \pi(k+1)) \\
(\text{since } \pi(k) = \pi(k+1) \text{ can never hold (because } \pi \in S_n)) \\
(k \notin \text{Des } \pi).
\end{pmatrix}
\]

In other words, the elements of $\text{Des}(w_0 \circ \pi)$ are precisely the elements of $[n-1]$ that don’t belong to $\text{Des } \pi$. In other words, the set $\text{Des}(w_0 \circ \pi)$ is the complement of the set $\text{Des } \pi$ in $[n-1]$. Thus,

\[
|\text{Des}(w_0 \circ \pi)| = |[n-1]| - |\text{Des } \pi| = n-1 - |\text{Des } \pi|.
\]

Now, forget that we fixed $\pi$. We thus have proven (3) for each $\pi \in S_n$.

Now, let $\pi \in D_{n,p}$. Then, $\pi$ has exactly $p$ descents. In other words, $|\text{Des } \pi| = p$. Thus, (3) yields $|\text{Des}(w_0 \circ \pi)| = n-1 - |\text{Des } \pi| = n-1 - p$. In other words, the permutation $w_0 \circ \pi$ has exactly $n-1-p$ descents. In other words, $w_0 \circ \pi \in D_{n,n-1-p}$ (since the definition of $D_{n,n-1-p}$ yields $D_{n,n-1-p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } n-1-p \text{ descents}\}$).

Now, forget that we fixed $\pi$. We thus have proven that $w_0 \circ \pi \in D_{n,n-1-p}$ for each $\pi \in D_{n,p}$. Thus, the map

\[
D_{n,p} \rightarrow D_{n,n-1-p}, \\
\pi \mapsto w_0 \circ \pi
\]

is well-defined. The same argument (but with $p$ replaced by $n-1-p$) shows that the map

\[
D_{n,n-1-p} \rightarrow D_{n,n-1-(n-1-p)}, \\
\pi \mapsto w_0 \circ \pi
\]

is well-defined. In other words, the map

\[
D_{n,n-1-p} \rightarrow D_{n,p}, \\
\pi \mapsto w_0 \circ \pi
\]

is well-defined (since $n-1-(n-1-p) = p$). But $w_0 \circ w_0 = \text{id}$ (since each $k \in [n]$ satisfies $(w_0 \circ w_0)(k) = w_0(w_0(k)) = n+1-n-1-k = k = \text{id}(k)$)

Thus, the two maps (4) and (5) are mutually inverse. Hence, these two maps are bijections. Thus, we have found a bijection from $D_{n,p}$ to $D_{n,n-1-p}$. Hence, $|D_{n,p}| = |D_{n,n-1-p}|$. This solves the exercise.

\footnote{since $\pi \in D_{n,p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents}\}$}
3.3 Remark

1. A similar solution could have been obtained by using the permutation \( \pi \circ w_0 \) instead of \( w_0 \circ \pi \). Indeed, similarly to (3), we also have

\[
|\text{Des} (\pi \circ w_0)| = n - 1 - |\text{Des} \pi| \quad \text{for each } \pi \in S_n.
\]

To prove this, we would have to show that

\[
\text{Des} (\pi \circ w_0) = \{n - k \mid k \in [n - 1] \setminus \text{Des} \pi\}
\]

(which is only a tad more complicated than proving that \( \text{Des} (w_0 \circ \pi) = [n - 1] \setminus \text{Des} \pi \)).

2. I have snuck a correction into the exercise: It used to only require \( n \in \mathbb{N} \), but now it requires \( n \) to be a positive integer. Indeed, the claim fails for \( n = 0 \). Sorry!

4 Exercise 4

4.1 Problem

Let \( n \in \mathbb{N} \). Let \( S = \{s_1 < s_2 < \cdots < s_k\} \) be a subset of \([n - 1]\). Set \( s_0 = 0 \) and \( s_{k+1} = n \). For each \( i \in [k+1] \), set \( d_i = s_i - s_{i-1} \). (You might remember this construction from the definition of the map \( D \) in the solution to Exercise 1 on homework set #0.)

(a) Prove that

\[
|\{\sigma \in S_n \mid \text{Des} \sigma \subseteq S\}| = \binom{n}{d_1, d_2, \ldots, d_{k+1}}.
\]

(The term on the right hand side is a multinomial coefficient. The \( \text{Des} \sigma \) on the left hand side denotes the descent set of \( \sigma \), that is, the set of all descents of \( \sigma \).)

(b) Prove that

\[
|\{\sigma \in S_n \mid \text{Des} \sigma = S\}| = \sum_{T \subseteq S} (-1)^{|S|-|T|} |\{\sigma \in S_n \mid \text{Des} \sigma \subseteq T\}|.
\]

4.2 Solution sketch

(a) A permutation \( \sigma \in S_n \) satisfies \( \text{Des} \sigma \subseteq S \) if and only if it is strictly increasing on each of the \( k + 1 \) intervals

\[
[s_0 + 1, s_1], \quad [s_1 + 1, s_2], \quad [s_2 + 1, s_3], \quad \ldots, \quad [s_k + 1, s_{k+1}].
\]

Hence, a permutation \( \sigma \in S_n \) satisfying \( \text{Des} \sigma \subseteq S \) is uniquely determined by the images

\[
\sigma ([s_0 + 1, s_1]), \quad \sigma ([s_1 + 1, s_2]), \quad \sigma ([s_2 + 1, s_3]), \quad \ldots, \quad \sigma ([s_k + 1, s_{k+1}])
\]

of these \( k + 1 \) intervals (indeed, once these images are known, we can use the strict increasingness of \( \sigma \) on these intervals to reconstruct each value of \( \sigma \)). These images must be disjoint subsets of \([n]\) (since \( \sigma \) is injective) and have the same sizes as the \( k + 1 \) intervals themselves (for the same reason); these sizes are

\[
s_1 - s_0 = d_1, \quad s_2 - s_1 = d_2, \quad s_3 - s_2 = d_3, \quad \ldots, \quad s_{k+1} - s_k = d_{k+1}.
\]

Thus, every permutation \( \sigma \in S_n \) satisfying \( \text{Des} \sigma \subseteq S \) can be constructed by the following algorithm:
• We choose a \(d_1\)-element subset of \([n]\) to be the image \(\sigma([s_0 + 1, s_1])\). This subset can be chosen in \(\binom{n}{d_1}\) ways.

• Next, we choose a \(d_2\)-element subset of \([n]\) to be the image \(\sigma([s_1 + 1, s_2])\), requiring that it be disjoint from the already chosen subset \(\sigma([s_0 + 1, s_1])\). This subset can be chosen in \(\binom{n - d_1}{d_2}\) ways (because by requiring it to be disjoint from the \(d_1\)-element subset \(\sigma([s_0 + 1, s_1])\), we are forcing it to be a \(d_2\)-element subset of the \((n - d_1)\)-element set \([n]\ \setminus \sigma([s_0 + 1, s_1])\)).

• Next, we choose a \(d_3\)-element subset of \([n]\) to be the image \(\sigma([s_2 + 1, s_3])\), requiring that it be disjoint from the already chosen subsets \(\sigma([s_0 + 1, s_1])\) and \(\sigma([s_1 + 1, s_2])\). This subset can be chosen in \(\binom{n - d_1 - d_2}{d_3}\) ways (because by requiring it to be disjoint from the \(d_1\)-element subset \(\sigma([s_0 + 1, s_1])\) and the \(d_2\)-element subset \(\sigma([s_1 + 1, s_2])\), we are forcing it to be a \(d_3\)-element subset of the \((n - d_1 - d_2)\)-element set \([n]\ \setminus \sigma([s_0 + 1, s_1])\ \setminus \sigma([s_1 + 1, s_2])\))\(^2\).

• And so on, until all \(k + 1\) images

\[
\sigma([s_0 + 1, s_1]), \quad \sigma([s_1 + 1, s_2]), \quad \sigma([s_2 + 1, s_3]), \quad \ldots, \quad \sigma([s_k + 1, s_{k+1}])
\]

are chosen. As we know, at this point, \(\sigma\) is uniquely determined.

The total number of ways in which this construction can be carried out is

\[
\binom{n}{d_1}\binom{n - d_1}{d_2}\binom{n - d_1 - d_2}{d_3}\ldots\binom{n - d_1 - d_2 - \cdots - d_k}{d_{k+1}}
\]

\[
= \prod_{i=0}^{k} \binom{n - d_1 - d_2 - \cdots - d_i}{d_{i+1}} = \prod_{i=1}^{k+1} \binom{n - d_1 - d_2 - \cdots - d_{i-1}}{d_i} = \binom{n}{d_1, d_2, \ldots, d_{k+1}}
\]

(by the first equation in Proposition 2.38 in the class notes (2018-10-03)). Thus, the number of permutations \(\sigma \in S_n\) satisfying \(\text{Des} \sigma \subseteq S\) is \(\binom{n}{d_1, d_2, \ldots, d_{k+1}}\). This solves part (a) of the exercise.

(b) We need the following result:

**Proposition 4.1.** Let \(G\) be a finite set. Let \(S\) be a subset of \(G\). Then,

\[
\sum_{I \subseteq G: \ |S| \leq I} (-1)^{|I|} = (-1)^{|S|} [G = S].
\]

Proposition 4.1 was proven during the solution of Exercise 6 on homework set #3.

\(^2\)Of course, we are tacitly using the fact that the two already chosen subsets \(\sigma([s_0 + 1, s_1])\) and \(\sigma([s_1 + 1, s_2])\) are disjoint (so that the set \([n] \setminus \sigma([s_0 + 1, s_1]) \setminus \sigma([s_1 + 1, s_2])\) really a \((n - d_1 - d_2)\)-element set).
We have

\[
\sum_{\sigma \in S_n} (-1)^{|S|-|T|} \{\{\text{Des } \sigma \sqsubseteq T\} | \text{Des } \sigma \sqsubseteq I\}
\]

\[
= \sum_{\sigma \in S_n} (-1)^{|S|-|I|} \{\{\text{Des } \sigma \sqsubseteq I\}\}
\]

(we have renamed the summation index \(T\) as \(I\))

\[
= \sum_{\sigma \in S_n} (-1)^{|S|-|I|} \sum_{U \subseteq I} \{\{\text{Des } \sigma = U\}\}
\]

\[
= \sum_{\sigma \in S_n} \sum_{U \subseteq I} (-1)^{|S|-|I|} \{\{\text{Des } \sigma = U\}\}
\]

\[
= \sum_{\sigma \in S_n} \sum_{U \subseteq I} (-1)^{|S|} (-1)^{|I|} \{\{\text{Des } \sigma = U\}\}
\]

\[
= \sum_{U \subseteq S} \left( \sum_{\sigma \in S_n} (-1)^{|I|} \right) (-1)^{|S|} \{\{\text{Des } \sigma = U\}\}
\]

(by Proposition 4.1 applied to \(S\) and \(U\) instead of \(G\) and \(S\))

\[
= \sum_{U \subseteq S} (-1)^{|U|} [S = U] (-1)^{|S|} \{\{\text{Des } \sigma = U\}\}
\]

\[
= \sum_{U \subseteq S; U \neq S} (-1)^{|U|} [S = U] (-1)^{|S|} \{\{\text{Des } \sigma = U\}\}
\]

(since \(U \neq S\))

\[
+ (-1)^{|S|} [S = S] (-1)^{|S|} \{\{\text{Des } \sigma = S\}\}
\]

(here, we have split off the addend for \(U = S\) from the sum)

\[
= \sum_{U \subseteq S; U \neq S} (-1)^{|U|} 0 (-1)^{|S|} \{\{\text{Des } \sigma = U\}\}
\]

\[
+ (-1)^{|S|} [S = S] (-1)^{|S|} \{\{\text{Des } \sigma = S\}\}
\]

\[
= (-1)^{|S|} (-1)^{|S|} \{\{\text{Des } \sigma = S\}\}
\]

This solves part (b) of the exercise.
5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. We shall follow the convention that $t_{i,i}$ denotes the identity permutation $id \in S_n$ for each $i \in [n]$.

Let $\sigma \in S_n$.

It is known that there is a unique $n$-tuple $(i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]$ satisfying $\sigma = t_{i_1,i_1} \circ t_{i_2,i_2} \circ \cdots \circ t_{i_n,i_n}$. (See [Grinbe16, Exercise 5.9] for the proof of this fact, or – easier – do it on your own.) Consider this $n$-tuple. (It is sometimes called the transposition code of $\sigma$.)

For each $k \in \{0, 1, \ldots, n\}$, we define a permutation $\sigma_k \in S_n$ by $\sigma_k = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}$.

Note that this permutation $\sigma_k$ leaves each of the numbers $k+1, k+2, \ldots, n$ unchanged (since all of $i_1, i_2, \ldots, i_k$, as well as $1, 2, \ldots, k$, are $\leq k$).

For each $k \in [n]$, let $m_k = \sigma_k(k)$.

(a) Show that $m_k \in [k]$ for all $k \in [n]$.

(b) Show that $\sigma_k(i_k) = k$ for all $k \in [n]$.

(c) Show that $\sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}$.

(d) Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be any $2n$ numbers. Prove that

$$\sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} x_k y_{\sigma(k)} = \sum_{k=1}^{n} (x_{i_k} - x_k) (y_{m_k} - y_k).$$

(e) Now assume that the numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real and satisfy $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Conclude that

$$\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\sigma(k)}.$$

5.2 Remark

This exercise is part of [Grinbe16, Exercise 5.25].

Parts (a) and (c), combined, show that $(m_1, m_2, \ldots, m_n)$ is the transposition code of $\sigma^{-1}$.

Part (e) of the exercise is known as the rearrangement inequality. The proof in this exercise is far from its easiest proof, but has the advantage of “manifest positivity” – i.e., it gives an explicit formula for the difference between the two sides as a sum of products of nonnegative numbers.

5.3 Solution Sketch

Let us first notice that any two elements $u, v \in [n]$ and any permutation $\pi \in S_n$ satisfy

$$t_{\pi(u),\pi(v)} \circ \pi = \pi \circ t_{u,v}. \quad (6)$$

[Proof of (6): Let $u, v \in [n]$ and $\pi \in S_n$. Fix $k \in [n]$. We shall prove that $(t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k)$.

Indeed, we are in one of the following three cases:
Case 1: We have \( k = u \).
Case 2: We have \( k = v \).
Case 3: We have neither \( k = u \) nor \( k = v \).

Let us first consider Case 1. In this case, we have \( k = u \). Thus, \( t_{u,v}(k) = t_{u,v}(u) = v \) (independently of whether \( u = v \) or \( u \neq v \)). Also, from \( k = u \), we obtain
\[
(t_{\pi(u),\pi(v)} \circ \pi)(k) = (t_{\pi(u),\pi(v)} \circ \pi)(u) = t_{\pi(u),\pi(v)}(\pi(u)) = \pi(v)
\]
(again, independently of whether \( \pi(u) = \pi(v) \) holds or not). Comparing this with
\[
(\pi \circ t_{u,v})(k) = \pi(t_{u,v}(k)) = \pi(v) \quad \text{(since \( t_{u,v}(k) = v \)),}
\]
we obtain \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \). Hence, \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \) is proven in Case 1.

The argument in Case 2 is analogous, and we leave it to the reader.

Let us now consider Case 3. In this case, we have neither \( k = u \) nor \( k = v \). Thus, \( t_{u,v}(k) \neq k \) (independently of whether \( u = v \) or \( u \neq v \)). Also, recall that we have neither \( k = u \) nor \( k = v \). Thus, we have neither \( \pi(k) = \pi(u) \) nor \( \pi(k) = \pi(v) \) (since the map \( \pi \) is injective (because \( \pi \in S_n \)). Hence, \( t_{\pi(u),\pi(v)}(\pi(k)) = \pi(k) \) (again, independently of whether \( \pi(u) = \pi(v) \) holds or not). Now,
\[
(t_{\pi(u),\pi(v)} \circ \pi)(k) = t_{\pi(u),\pi(v)}(\pi(k)) = \pi(k).
\]
Comparing this with
\[
(\pi \circ t_{u,v})(k) = \pi(t_{u,v}(k)) = \pi(k) \quad \text{(since \( t_{u,v}(k) = k \)),}
\]
we obtain \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \). Hence, \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \) is proven in Case 3.

We have now proven \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \) in each of the three Cases 1, 2 and 3. Thus, \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \) always holds.

Forget now that we fixed \( k \). We thus have shown that \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \) for each \( k \in [n] \). In other words, \( t_{\pi(u),\pi(v)} \circ \pi = \pi \circ t_{u,v} \). Thus, \( (6) \) is proven.

Recall that \((i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n] \). Thus,
\[
i_j \in [j] \quad \text{for each } j \in [n]. \tag{7}
\]

The definition of \( \sigma_0 \) shows that
\[
\sigma_0 = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{0,i_0} = \text{(composition of 0 permutations)} = id.
\]

The definition of \( \sigma_n \) shows that
\[
\sigma_n = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n} = \sigma.
\]

\[\text{(a)}\] Let \( k \in [n] \). Then, from (7), we conclude that each \( j \in [k] \) satisfies \( i_j \in [j] \subseteq [k] \) (since \( j \leq k \)). Hence, the \( k \) numbers \( i_1, i_2, \ldots, i_k \) all belong to \( [k] \). The same holds for the \( k \) numbers \( 1, 2, \ldots, k \). Thus, the \( k \) permutations \( t_{1,i_1}, t_{2,i_2}, \ldots, t_{k,i_k} \) all preserve the set \( [k] \).
\[
\text{Hence, their composition } t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} \text{ preserves the set } [k] \text{ as well.} \tag{8}
\]

\[\text{We say that a map } \tau : [n] \rightarrow [n] \text{ preserves a subset } S \text{ of } [n] \text{ if and only if it satisfies } \tau(S) \subseteq S. \text{ This does not mean that } \tau(s) = s \text{ for each } s \in S; \text{ it only means that } \tau \text{ sends each element of } S \text{ to a (possibly different) element of } S. \]

\[\text{Here, we are using the following fact: If } S \text{ is a subset of } [n], \text{ and if } \alpha_1, \alpha_2, \ldots, \alpha_k \text{ are } k \text{ maps from } [n] \text{ to } [n] \text{ that all preserve the set } S, \text{ then the composition } \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k \text{ of these } k \text{ maps must preserve the set } S \text{ as well.} \text{ (This is easy to prove by induction on } k.\)
\[ \sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k}, \] this rewrites as follows: The map \( \sigma_k \) preserves the set \([k]\). In other words, \( \sigma_k ([k]) \subseteq [k] \). Now, \( k \in [k] \), so that \( \sigma_k (k) \in \sigma_k ([k]) \subseteq [k] \). Hence, \( m_k = \sigma_k (k) \in [k] \). This solves part (\( a \)) of the exercise.

(b) Let \( k \in [n] \). Then, from (7), we conclude that each \( j \in [k - 1] \) satisfies \( i_j \in [j] \subseteq [k - 1] \) (since \( j \leq k - 1 \)). Hence, the \( k - 1 \) numbers \( i_1, i_2, \ldots, i_{k-1} \) all belong to \([k - 1] \). The same holds for the \( k - 1 \) numbers \( 1, 2, \ldots, k - 1 \). Thus, the \( k - 1 \) permutations \( t_{1,i_1}, t_{2,i_2}, \ldots, t_{k-1,i_{k-1}} \) all leave each of the numbers \( k, k + 1, \ldots, n \) unchanged. Hence, their composition \( t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}} \) leaves each of the numbers \( k, k + 1, \ldots, n \) unchanged. In particular, it thus leaves the number \( k \) unchanged. In other words,

\[ (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) (k) = k. \]

The definition of \( \sigma_k \) yields

\[ \sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ t_{k,i_k}. \]

Hence,

\[
\sigma_k (i_k) = \left( (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ t_{k,i_k} \right) (i_k) = \left( (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ (t_{k,i_k} (i_k)) \right) = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) (k) = k.
\]

This solves part (b) of the exercise.

(c) We shall prove that

\[ \sigma_p^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{p,m_p} \quad \text{for each } p \in \{0, 1, \ldots, n\}. \]

[Proof of (8): We shall prove (8) by induction on \( p \):

Induction base: In the case of \( p = 0 \), the equality (8) holds, since \( \sigma_0 \) is defined as an empty composition whereas the right hand side of (8) also is an empty composition in this case. This completes the induction base.

Induction step: Let \( k \in [n] \). Assume that (8) holds for \( p = k - 1 \). We must prove that (8) holds for \( p = k \).

We have assumed that (8) holds for \( p = k - 1 \). That is, we have

\[ \sigma_{k-1}^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{k-1,m_{k-1}}. \]

Part (b) of the exercise yields \( \sigma_k (i_k) = k \), whereas the definition of \( m_k \) yields \( \sigma_k (k) = m_k \). But (6) (applied to \( \pi = \sigma_k, u = i_k \) and \( v = k \)) yields

\[ t_{\sigma_k(i_k),\sigma_k(k)} \circ \sigma_k = \sigma_k \circ t_{i_k,k}. \]

In view of \( \sigma_k (i_k) = k \) and \( \sigma_k (k) = m_k \), this rewrites as

\[ t_{k,m_k} \circ \sigma_k = \sigma_k \circ t_{i_k,k}. \]

We have \( \sigma_{k-1} = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}} \) (by the definition of \( \sigma_{k-1} \)). Now, the definition of \( \sigma_k \) yields

\[ \sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ t_{k,i_k} = \sigma_{k-1} \circ t_{k,i_k}. \]
Solving this equation for \( \sigma_{k-1} \), we obtain
\[
\sigma_{k-1} = \sigma_k \circ t_{k,3k}^{-1} = \sigma_k \circ t_{k,ik} = t_{k,m_k} \circ \sigma_k \quad \text{(by (8))}.
\] (11)

Solving this equation for \( \sigma_k \), we find
\[
\sigma_k = t_{k,m_k}^{-1} \circ \sigma_{k-1} = t_{k,m_k} \circ \sigma_{k-1}.
\]

Hence,
\[
\sigma_k^{-1} = (t_{k,m_k} \circ \sigma_{k-1})^{-1} = t_{k,m_k}^{-1} \circ \sigma_{k-1}^{-1} = t_{k,m_k}^{-1} \circ t_{k,m_k}^{-1} = t_{k,m_k}^{-1} \circ \sigma_{k-1}^{-1} = t_{k,m_k}^{-1} \circ \sigma_{k-1}^{-1} = t_{k,m_k}^{-1} \circ \sigma_{k-1}^{-1}.
\]

In other words, (8) holds for \( p = k \). This completes the induction step. Thus, (8) is proven by induction.

Applying (8) to \( p = n \), we obtain \( \sigma_n^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n} \). In view of \( \sigma_n = \sigma \), this rewrites as \( \sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n} \). This solves part (c) of the exercise.

(d) For each permutation \( \tau \in S_n \), we define a number \( z(\tau) \) by
\[
z(\tau) = \sum_{k=1}^{n} x_k y_{\tau(k)}.
\]

We shall show that
\[
z(\sigma_{p-1}) - z(\sigma_p) = (x_{i_p} - x_p) \ (y_{m_p} - y_p) \quad \text{for each} \ p \in [n].
\] (12)

Proof of (12): Let \( p \in [n] \). Applying (10) to \( k = p \), we obtain \( \sigma_p = \sigma_{p-1} \circ t_{p,i_p} \). Hence, if \( p = i_p \), then (12) holds. Thus, for the rest of this proof, we WLOG assume that \( p \neq i_p \). Hence, \( t_{p,i_p} \) is an actual transposition (not the identity map).

From \( \sigma_p = \sigma_{p-1} \circ t_{p,i_p} \), we obtain
\[
\sigma_p(p) = (\sigma_{p-1} \circ t_{p,i_p})(p) = \sigma_{p-1}(t_{p,i_p}(p)) = \sigma_{p-1}(i_p),
\]
so that
\[
\sigma_{p-1}(i_p) = \sigma_p(p) = m_p
\] (since the definition of \( m_p \) yields \( m_p = \sigma_p(p) \)).

From \( \sigma_p = \sigma_{p-1} \circ t_{p,i_p} \), we also obtain
\[
\sigma_p(i_p) = (\sigma_{p-1} \circ t_{p,i_p})(i_p) = \sigma_{p-1}(t_{p,i_p}(i_p)) = \sigma_{p-1}(p),
\]
(13)

Proof. Assume that \( p = i_p \). Thus, \( i_p = p \), so that \( x_{i_p} - x_p = x_p - x_p = 0 \). Hence, the right hand side of (12) equals 0. Also, \( \sigma_p = \sigma_{p-1} \circ t_{p,i_p} = \sigma_{p-1} \), so that \( z(\sigma_{p-1}) - z(\sigma_p) = z(\sigma_{p-1}) - z(\sigma_{p-1}) = 0 \). Thus, the left hand side of (12) equals 0 as well. Hence, the equality (12) holds (since both its right hand side and its left hand side equal 0).
On the other hand, the definition of $\sigma_{p-1} (p) = \sigma_p (i_p) = p$ (by part (b) of the exercise, applied to $k = p$).

Every $k \in [n]$ satisfying $k \neq p$ and $k \neq i_p$ satisfies

$$\sigma_{p-1} (k) = \sigma_p (k)$$

Now, the definition of $z (\sigma_{p-1})$ yields

$$z (\sigma_{p-1}) = \sum_{k=1}^{n} x_k y_{\sigma_{p-1} (k)} = x_p y_{\sigma_{p-1} (p)} + x_{i_p} y_{\sigma_{p-1} (i_p)} + \sum_{k \in [n]; k \neq p \text{ and } k \neq i_p} x_k y_{\sigma_{p-1} (k)}$$

(here, we have split the addends for $k = p$ and for $k = i_p$ from the sum (and these are two distinct addends, since $p \neq i_p$)

$$= x_p y_p + x_{i_p} y_{m_p} + \sum_{k \in [n]; k \neq p \text{ and } k \neq i_p} x_k y_{\sigma_p (k)}.$$

On the other hand, the definition of $z (\sigma_p)$ yields

$$z (\sigma_p) = \sum_{k=1}^{n} x_k y_{\sigma_p (k)} = x_p y_{\sigma_p (p)} + x_{i_p} y_{\sigma_p (i_p)} + \sum_{k \in [n]; k \neq p \text{ and } k \neq i_p} x_k y_{\sigma_p (k)}$$

(here, we have split the addends for $k = p$ and for $k = i_p$ from the sum (and these are two distinct addends, since $p \neq i_p$)

$$= x_p y_{m_p} + x_{i_p} y_p + \sum_{k \in [n]; k \neq p \text{ and } k \neq i_p} x_k y_{\sigma_p (k)}.$$

Subtracting this equality from the preceding equality, we obtain

$$z (\sigma_{p-1}) - z (\sigma_p)$$

$$= \left( x_p y_p + x_{i_p} y_{m_p} + \sum_{k \in [n]; k \neq p \text{ and } k \neq i_p} x_k y_{\sigma_p (k)} \right) - \left( x_p y_{m_p} + x_{i_p} y_p + \sum_{k \in [n]; k \neq p \text{ and } k \neq i_p} x_k y_{\sigma_p (k)} \right)$$

$$= x_p y_p + x_{i_p} y_{m_p} - x_p y_{m_p} - x_{i_p} y_p = (x_{i_p} - x_p) (y_{m_p} - y_p).$$

This proves (12).

Proof: Let $k \in [n]$ be such that $k \neq p$ and $k \neq i_p$. Thus, $t_{p,i_p} (k) = k$. But $\sigma_p = \sigma_{p-1} \circ t_{p,i_p}$; hence,

$$\sigma_p (k) = (\sigma_{p-1} \circ t_{p,i_p}) (k) = \sigma_{p-1} \left( t_{p,i_p} (k) \right) = \sigma_{p-1} (k) = \sigma_p (k),$$

so that $\sigma_{p-1} (k) = \sigma_p (k)$, qed.
Now, the telescope principle yields
\[
\sum_{p=1}^{n} (z(\sigma_{p-1}) - z(\sigma_{p})) = z\left(\sigma_{0} = \text{id}\right) - z\left(\sigma_{n} = \sigma\right) = z(\text{id}) - z(\sigma) = \sum_{k=1}^{n} x_{k}y_{\sigma(k)} - \sum_{k=1}^{n} x_{k}y_{\sigma(k)} \quad (\text{by the definition of } z(\text{id})) \quad \text{and} \quad \sum_{k=1}^{n} x_{k}y_{\sigma(k)} \quad (\text{by the definition of } z(\sigma))
\]

Hence,
\[
\sum_{k=1}^{n} x_{k}y_{k} - \sum_{k=1}^{n} x_{k}y_{\sigma(k)} = \sum_{p=1}^{n} (z(\sigma_{p-1}) - z(\sigma_{p})) = \sum_{p=1}^{n} (x_{i_{p}} - x_{p}) (y_{m_{p}} - y_{p}) = \sum_{k=1}^{n} (x_{i_{k}} - x_{k}) (y_{m_{k}} - y_{k})
\]

(here, we have renamed the summation index \( p \) as \( k \)). This solves part (d) of the exercise.

(e) Fix \( k \in [n] \). Then, \( i_{k} \in [k] \) (by (7)), so that \( i_{k} \leq k \) and therefore \( x_{i_{k}} \geq x_{k} \) (since \( x_{1} \geq x_{2} \geq \cdots \geq x_{n} \)). Hence, \( x_{i_{k}} - x_{k} \geq 0 \).

Also, \( m_{k} \in [k] \) (by part (a) of the exercise), so that \( m_{k} \leq k \) and thus \( y_{m_{k}} \geq y_{k} \) (since \( y_{1} \geq y_{2} \geq \cdots \geq y_{n} \)). Hence, \( y_{m_{k}} - y_{k} \geq 0 \). Now,
\[
\left( x_{i_{k}} - x_{k} \right) \left( y_{m_{k}} - y_{k} \right) \geq 0.
\]

(16)

Now, forget that we fixed \( k \). We thus have proven (16) for each \( k \in [n] \). Now, part (d) of the exercise yields
\[
\sum_{k=1}^{n} x_{k}y_{k} - \sum_{k=1}^{n} x_{k}y_{\sigma(k)} = \sum_{k=1}^{n} (x_{i_{k}} - x_{k}) (y_{m_{k}} - y_{k}) \geq 0.
\]

(by 16)

In other words,
\[
\sum_{k=1}^{n} x_{k}y_{k} \geq \sum_{k=1}^{n} x_{k}y_{\sigma(k)}.
\]

This solves part (e) of the exercise.

6 Exercise 6

6.1 Problem

Prove the following:
(a) If \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \) are such that \( m < n \), then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)^m = 0.
\]

(b) If \( n \in \mathbb{N} \) and \( r \in [n - 1] \), then
\[
\sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n - k)^{2r} = 0.
\]

6.2 Solution sketch

(a) First solution to part (a): Let \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \) be such that \( m < n \). We have \( ||m|| = m < n = ||n|| \). Thus, there are no surjections from \( [m] \) to \( [n] \) (by the Pigeonhole Principle for Surjections). Recall that \( \text{sur} (m, n) \) denotes the number of all surjections from \( [m] \) to \( [n] \). Thus, \( \text{sur} (m, n) = 0 \) (since there are no surjections from \( [m] \) to \( [n] \)).

But Theorem 2.28 from class (2018-10-01) shows that
\[
\text{sur} (m, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^m = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)^m
\]
(here, we have renamed the summation index \( i \) as \( k \)). Comparing this with \( \text{sur} (m, n) = 0 \), we obtain
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - k)^m = 0.
\]
This solves part (a) of the exercise.

Second solution to part (a): Let \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \) be such that \( m < n \). Exercise 6 (b) on homework set #3 yields that if \( A_1, A_2, \ldots, A_n \) are \( n \) numbers, then
\[
\sum_{I \subseteq [n]} (-1)^{n-|I|} \left( \sum_{i \in I} A_i \right)^m = 0.
\]
Applying this to \( A_i = 1 \), we obtain
\[
\sum_{I \subseteq [n]} (-1)^{n-|I|} \left( \sum_{i \in I} 1 \right)^m = 0.
\]
Thus,

\[
0 = \sum_{I \subseteq [n]} (-1)^{|I|} \left( \sum_{i \in I \cap \{1, \ldots, |I|\}} \right)^m = \sum_{I \subseteq [n]} (-1)^{|I|} |I|^m
\]

= \sum_{i=0}^n \sum_{|I|=i} (-1)^{|I|-i} |I|^m = \sum_{i=0}^n (-1)^{n-i} m^i

= (the number of all \( I \subseteq [n] \) satisfying \(|I|=i\)) \cdot (-1)^{n-i} m^i

= \sum_{i=0}^n \left( \text{the number of all } I \subseteq [n] \text{ satisfying } |I|=i \right) \cdot (-1)^{n-i} m^i

= \sum_{i=0}^n \binom{n}{i} \cdot (-1)^{n-i} m^i = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} m^i = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (n-k)^m

(here, we have substituted \( n-k \) for \( i \) in the sum)

= \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m.

This solves part (a) of the exercise again.

Third solution to part (a): Part (a) of the exercise is a particular case of Theorem 6.1 further below (applied to \( a=m, b=n \) and \( c=n \)).

(b) We need a generalization of part (a) of the exercise:

**Theorem 6.1.** Let \( a \in \mathbb{N}, b \in \mathbb{Q} \) and \( c \in \mathbb{N} \) be such that \( c > a \). Then,

\[
\sum_{k=0}^c (-1)^k \binom{c}{k} (b-k)^a = 0.
\]

For the proof of Theorem 6.1, see [Grinbe18, Theorem 0.2].

Let \( n \in \mathbb{N} \) and \( r \in [n-1] \). We have \( r \in [n-1] \), thus \( r \leq n-1 \) and therefore \( 2r \leq 2(n-1) < 2n \). Thus, \( 2n > 2r \). Hence, Theorem 6.1 (applied to \( a=2r, b=n \) and \( c=2n \)) yields

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0.
\]
Thus,

\[
0 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (n - k)^{2r} \\
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n - k)^{2r} + \sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n - k)^{2r} \\
(\text{since } 0 \leq n \leq 2n). \text{ But}
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (n - k)^{2r} \\
= \sum_{k=0}^{n-1} (-1)^{2n-k} \binom{2n}{2n-k} \binom{n}{n - (2n - k)} (n - k)^{2r} \\
= \binom{2n}{k} \\
(\text{by the symmetry of Pascal’s triangle})
\]

(here, we have substituted $2n - k$ for $k$ in the sum)

\[
= \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (- (n - k))^{2r} = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (n - k)^{2r} \\
(\text{since } 2r \text{ is even})
\]

\[
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n - k)^{2r} - (-1)^n \binom{2n}{n} (n - n)^{2r} \\
(\text{here, we have extended the range of the sum to include a new addend for } k = n, \text{ and then subtracted that addend})
\]

\[
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n - k)^{2r}.
\]

Hence, (17) becomes

\[
0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n - k)^{2r} + \sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n - k)^{2r} \\
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} \\
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 2 \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}.
\]

Dividing this equality by 2, we find $0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}$. This solves part (b) of the exercise.

### 6.3 Remark

I have learnt part (b) of the exercise from [MathOverflow question \#312839](https://mathoverflow.net/questions/312839), which also asks if the sum is $\neq 0$ when $2r$ is replaced by an odd integer between 1 and $2n - 1$. 

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7 Exercise 7

7.1 Problem

Let \( n \in \mathbb{N} \) and \( d \in \mathbb{N} \). An \( n \)-tuple \( (x_1, x_2, \ldots, x_n) \in [d]^n \) is said to be all-even if each element of \([d]\) occurs an even number of times in this \( n \)-tuple (i.e., if for each \( k \in [d] \), the number of all \( i \in [n] \) satisfying \( x_i = k \) is even). For example, the 4-tuple \((1, 4, 4, 1)\) and the 6-tuples \((1, 3, 3, 5, 1, 5)\) and \((2, 4, 2, 4, 3, 3)\) are all-even, while the 4-tuples \((1, 2, 2, 4)\) and \((2, 4, 6, 4)\) are not.

Prove that the number of all all-even \( n \)-tuples \( (x_1, x_2, \ldots, x_n) \in [d]^n \) is

\[
\frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

[Hint: Compute the sum \( \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n \) in two ways. One way is to split it according to the number of \( i \in [d] \) satisfying \( e_i = -1 \); this is a number \( k \in \{0, 1, \ldots, d\} \). Another way is by using the product rule:

\[
(e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}
\]

and then simplifying each sum \( \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^n} e_{x_1} e_{x_2} \cdots e_{x_n} \) using a form of destructive interference. This is not unlike the number of 1-even \( n \)-tuples, which we computed at the end of [the 2018-10-10 class].]

7.2 Solution sketch

Recall the product rule (which we have already used when solving Exercise 6 on [homework set #3]):

**Proposition 7.1** (Product rule). Let \( m \in \mathbb{N} \). Let \( I \) be a finite set. Let \( P_{u,v} \), for all \( u \in [m] \) and \( v \in I \), be numbers or polynomials or square matrices of the same size. Then,

\[
\left( \sum_{i \in I} P_{1,i} \right) \left( \sum_{i \in I} P_{2,i} \right) \cdots \left( \sum_{i \in I} P_{m,i} \right) = \sum_{(i_1, i_2, \ldots, i_m) \in I^m} P_{1,i_1} P_{2,i_2} \cdots P_{m,i_m}.
\]

Fix a \( d \)-tuple \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d \). We now apply Proposition 7.1 to \( m = n \), \( I = [d] \) and \( P_{u,v} = e_v \). As a result, we obtain

\[
\left( \sum_{i \in [d]} e_i \right) \left( \sum_{i \in [d]} e_i \right) \cdots \left( \sum_{i \in [d]} e_i \right) = \sum_{(i_1, i_2, \ldots, i_n) \in [d]^n} e_{i_1} e_{i_2} \cdots e_{i_n} = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}
\]

n times
(here, we have renamed the summation index \((i_1, i_2, \ldots, i_n)\) as \((x_1, x_2, \ldots, x_n)\)). Thus,

\[
\sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n} = \left( \sum_{i \in [d]} e_i \right) \left( \sum_{i \in [d]} e_i \right) \cdots \left( \sum_{i \in [d]} e_i \right) = \left( \sum_{i \in [d]} e_i \right)^n
\]

\[
= (e_1 + e_2 + \cdots + e_d)^n.
\]  

(18)

Now, forget that we fixed \((e_1, e_2, \ldots, e_d)\). We thus have proven the equality (18) for each \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d\).

Now,

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}
\]  

(by (18))

\[
= \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}
\]

\[
= \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}.
\]  

(19)

We shall now simplify the inner sum on the right hand side of this equality. Indeed, we claim the following:

**Claim 1:** Let \((x_1, x_2, \ldots, x_n) \in [d]^n\).

(a) If the \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is not all-even, then

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = 0.
\]

(b) If the \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is all-even, then

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = 2^d.
\]

**Proof of Claim 1:** (a) Assume that the \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is not all-even. Thus, it is **not** true that for each \(k \in [d]\), the number of all \(i \in [n]\) satisfying \(x_i = k\) is even (by the definition of “all-even”). In other words, there exists some \(k \in [d]\) such that the number of all \(i \in [n]\) satisfying \(x_i = k\) is odd. Consider this \(k\).

The number

\[
\sum_{i \in [n]} [x_i = k] = \sum_{i \in [n]; x_i = k} [x_i = k] + \sum_{i \in [n]; x_i \neq k} [x_i = k] = \sum_{i \in [n]; x_i = k} 1 + \sum_{i \in [n]; x_i \neq k} 0
\]

\[
= \sum_{i \in [n]; x_i = k} 1 = (\text{the number of all } i \in [n] \text{ satisfying } x_i = k) \cdot 1
\]

= (the number of all \(i \in [n]\) satisfying \(x_i = k\))

\[
= (\text{the number of all } i \in [n] \text{ satisfying } x_i = k)
\]
is odd (by the definition of $k$). Now,
\[
(-1)^{[x_1=k]} (-1)^{[x_2=k]} \cdots (-1)^{[x_n=k]} = \prod_{i \in [n]} (-1)^{[x_i=k]} = (-1)^{\sum_{i \in [n]} [x_i=k]} = -1
\]
(since the number $\sum_{i \in [n]} [x_i = k]$ is odd).

Now, define the two subsets
\[
N = \{ (e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d \mid e_k = -1 \}
\quad \text{and}
\]
\[
P = \{ (e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d \mid e_k = 1 \}
\]
of the set $\{-1, 1\}^d$. Clearly, each element of $\{-1, 1\}^d$ belongs to exactly one of these two subsets $N$ and $P$ (because for each $(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d$, we have either $e_k = -1$ or $e_k = 1$ but not both).

Clearly, the map
\[
N \to P, \quad (e_1, e_2, \ldots, e_d) \mapsto (e_1, e_2, \ldots, e_k-1, -e_k, e_{k+1}, e_{k+2}, \ldots, e_d)
\]
(which replaces the $k$-th entry of a $d$-tuple by its negative, while leaving all other entries unchanged) is well-defined and bijective (indeed, its inverse map is defined by the same rule). We can rewrite this map (using the Iverson bracket notation) as the map
\[
N \to P, \quad (e_1, e_2, \ldots, e_d) \mapsto \left( (-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d \right)
\]
(because each $(e_1, e_2, \ldots, e_d) \in N$ satisfies
\[
\left( (-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d \right) = (e_1, e_2, \ldots, e_k-1, -e_k, e_{k+1}, e_{k+2}, \ldots, e_d)
\]
\footnotemark[7]. Hence, the map
\[
N \to P, \quad (e_1, e_2, \ldots, e_d) \mapsto \left( (-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d \right)
\]
is bijective, i.e., is a bijection from $N$ to $P$.

\footnotetext[7]{Proof. Let $(e_1, e_2, \ldots, e_d) \in N$. Then, each $i \in [d]$ satisfying $i \neq k$ satisfies $i = k$ = 0 and therefore $(-1)^{[i=k]} e_i = (-1)^0 e_i = e_i$. Hence, the $d$-tuple $\left( (-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d \right)$ differs from the $d$-tuple $(e_1, e_2, \ldots, e_d)$ only in its $k$-th entry. As for its $k$-th entry, it is $(-1)^{[k=k]} e_k = -e_k$. Thus, this $d$-tuple $\left( (-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d \right)$ is obtained from the $d$-tuple $(e_1, e_2, \ldots, e_d)$ by replacing its $k$-th entry by $-e_k$. In other words,
\[
\left( (-1)^{[1=k]} e_1, (-1)^{[2=k]} e_2, \ldots, (-1)^{[d=k]} e_d \right) = (e_1, e_2, \ldots, e_{k-1}, -e_k, e_{k+1}, e_{k+2}, \ldots, e_d)\].}
Recall that each element of $\{-1,1\}^d$ belongs to exactly one of the two subsets $N$ and $P$. Hence, we can split the sum $\sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}$ as follows:

$$
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} + \sum_{(e_1, e_2, \ldots, e_d) \in P} e_{x_1} e_{x_2} \cdots e_{x_n}
$$

$$
= \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} + \sum_{(e_1, e_2, \ldots, e_d) \in P} (\prod_{i=1}^{d} e_{x_i}) (\prod_{i=1}^{k} e_{x_i}) (\prod_{i=k+1}^{d} e_{x_i})
$$

$$
= \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} + \sum_{(e_1, e_2, \ldots, e_d) \in P} (\prod_{i=1}^{d} e_{x_i}) (\prod_{i=1}^{k} e_{x_i}) (\prod_{i=k+1}^{d} e_{x_i})
$$

$$
= \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} - \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} = 0.
$$

This proves Claim 1 (a).

(b) Assume that the $n$-tuple $(x_1, x_2, \ldots, x_n)$ is all-even. Thus, for each $k \in [d]$, the number of all $i \in [n]$ satisfying $x_i = k$ is even (by the definition of “all-even”).

Let $k \in [d]$. As we have just seen, the number of all $i \in [n]$ satisfying $x_i = k$ is even. In other words, there exists some $h \in \mathbb{Z}$ such that

$$
\text{(the number of all } i \in [n] \text{ satisfying } x_i = k) = 2h. \quad (20)
$$

Consider this $h$.

Now, let $(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d$ be arbitrary. Thus, $e_k \in \{-1,1\}$, so that $e_k^2 = 1$. Now,

$$
\prod_{i \in [n]; \ x_i = k} e_{x_i} = \prod_{i \in [n]; \ x_i = k} e_k \text{ (the number of all } i \in [n] \text{ satisfying } x_i = k) = e_k^{2h} \quad \text{(by (20))}
$$

$$
= \left( e_k^{2} \right)^h = 1^h = 1. \quad (21)
$$

Now, forget that we fixed $(e_1, e_2, \ldots, e_d)$ and $k$. We thus have proven (21) for each $(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d$ and $k \in [d]$.

Now, each $(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d$ satisfies

$$
e_{x_1} e_{x_2} \cdots e_{x_n} = \prod_{i \in [n]} e_{x_i} = \prod_{k \in [d]} \prod_{i \in [n]; \ x_i = k} e_{x_i} = \prod_{k \in [d]} 1 = 1. \quad \text{(by (21))}
$$
Hence,
\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} \underbrace{e_{x_1}e_{x_2} \cdots e_{x_n}}_{=1} = \sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} 1 = \left|\{-1,1\}^d\right| \cdot 1
\]
\[
= \left|\{-1,1\}^d\right| = \left|\{-1,1\}\right|^d = 2^d.
\]

This proves Claim 1 (b).]

Now, (19) becomes
\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} (e_1 + e_2 + \cdots + e_d)^n
\]
\[
= \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} \sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} e_{x_1}e_{x_2} \cdots e_{x_n}
\]
\[
= \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n; (x_1, x_2, \ldots, x_n) \text{ is all-even}} \sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} e_{x_1}e_{x_2} \cdots e_{x_n}
\]
\[
= 2^d
\]
(by Claim 1 (b))

\[
+ \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n; (x_1, x_2, \ldots, x_n) \text{ is not all-even}} \sum_{(e_1, e_2, \ldots, e_d) \in \{-1,1\}^d} e_{x_1}e_{x_2} \cdots e_{x_n}
\]
\[
= \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n; (x_1, x_2, \ldots, x_n) \text{ is all-even}} 2^d
\]
= (the number of all all-even \( (x_1, x_2, \ldots, x_n) \in [d]^n \) \) \[22\]

For each \( d \)-tuple \( (e_1, e_2, \ldots, e_d) \in \{-1,1\}^d \), we have
\[
d - (e_1 + e_2 + \cdots + e_d) = \frac{d}{\sum_{i \in [d]} {1 - e_i}} \sum_{i \in [d]} {e_i} = \sum_{i \in [d]} {1 - e_i} = \sum_{i \in [d]} (1 - e_i)
\]
\[
= \sum_{i \in [d]; e_i = -1} \left(1 - \frac{e_i}{-1}\right) + \sum_{i \in [d]; e_i = 1} \left(1 - \frac{e_i}{1}\right)
\]
\[
= \sum_{i \in [d]; e_i = -1} \left(1 - (-1)) \right) + \sum_{i \in [d]; e_i = 1} \left(1 - 1\right)
\]
\[
= \sum_{i \in [d]; e_i = -1} 2 + \sum_{i \in [d]; e_i = 1} 0
\]
\[
= \sum_{i \in [d]; e_i = -1} 2 = \left|\{i \in [d] \mid e_i = -1\}\right| \cdot 2 = 2 \cdot \left|\{i \in [d] \mid e_i = -1\}\right|
\]

and thus
\[
e_1 + e_2 + \cdots + e_d = d - 2 \cdot \left|\{i \in [d] \mid e_i = -1\}\right|.
\]
Comparing this with (22), we obtain

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} \left( e_1 + e_2 + \cdots + e_d \right)^n = \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n.
\]  

(23)

Hence, (23) becomes

\[
2^n \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

On the other hand, a \(d\)-tuple \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d\) is uniquely determined by the set \(\{i \in [d] \mid e_i = -1\}\) of all positions at which it contains a \(-1\) (and conversely, for every subset \(S\) of \([d]\), there exists such a \(d\)-tuple whose set \(\{i \in [d] \mid e_i = -1\}\) is \(S\)). Thus, the map

\[
\{-1, 1\}^d \to \{S \subseteq [d]\}, \quad (e_1, e_2, \ldots, e_d) \mapsto \{i \in [d] \mid e_i = -1\}
\]

is a bijection. Hence, we can substitute \(S\) for \(\{i \in [d] \mid e_i = -1\}\) in the sum on the right hand side of (23). We thus obtain

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n
= \sum_{S \subseteq [d]} (d - 2 \cdot |S|)^n = \sum_{k=0}^{d} \sum_{S \subseteq [d]; |S| = k} (d - 2 \cdot |S|)^n
= \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

Hence, (23) becomes

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} \left( e_1 + e_2 + \cdots + e_d \right)^n
= \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n
= \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

Comparing this with (22), we obtain

\[
\text{(the number of all all-even } (x_1, x_2, \ldots, x_n) \in [d]^n\} \cdot 2^d = \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]
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Solving this for (the number of all all-even \((x_1, x_2, \ldots, x_n) \in [d]^n\)), we obtain

\[
\text{(the number of all all-even } (x_1, x_2, \ldots, x_n) \in [d]^n) = \frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

This solves the exercise.

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The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10.

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