1 Exercise 1

1.1 Problem
Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Let $i$ and $j$ be two elements of $[n]$ such that $i < j$ and $\sigma(i) > \sigma(j)$. Let $Q$ be the set of all $k \in \{i+1, i+2, \ldots, j-1\}$ satisfying $\sigma(i) > \sigma(k) > \sigma(j)$. Prove that

$$\ell(\sigma \circ t_{i,j}) = \ell(\sigma) - 2|Q| - 1.$$ 

1.2 Remark
This exercise implies that, in particular, $\ell(\sigma \circ t_{i,j}) < \ell(\sigma)$; this answers the question on page 213 of the notes from class (2018-10-22).

1.3 Solution

[...]
2 Exercise 2

2.1 Problem

Let \( n \in \mathbb{N} \) and \( \pi \in S_n \).

(a) Prove that
\[
\sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j).
\]

(b) Prove that
\[
\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j - i).
\]

2.2 Solution

We shall use the following fact:

**Proposition 2.1.** Let \( n \in \mathbb{N} \). Let \( \sigma \in S_n \). Let \( a_1, a_2, \ldots, a_n \) be any \( n \) numbers. (Here, “number” means “real number” or “complex number” or “rational number”, as you prefer; this makes no difference.) Prove that
\[
\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)} (a_j - a_i) = \sum_{i=1}^{n} a_i (i - \sigma(i)).
\]

[Here, the summation sign “\( \sum \)” means “\( \sum_{(i,j) \in \{1,2,\ldots,n\}^2; i<j \text{ and } \sigma(i) > \sigma(j)} \); this is a sum over all inversions of \( \sigma \).]

Proposition 2.1 is [Grinbe16, Exercise 5.19]. For a different proof of it, see [Gorski18, Exercise 4].

Now, let us solve the exercise. We have \( \pi \in S_n \). In other words, \( \pi \) is a permutation of \([n]\). In other words, \( \pi \) is a bijection \([n] \rightarrow [n]\). Hence, we can substitute \( \pi(i) \) for \( i \) in the sum \( \sum_{i \in [n]} i^2 \). We thus obtain
\[
\sum_{i \in [n]} i^2 = \sum_{i \in [n]} (\pi(i))^2. \tag{1}
\]

(a) Proposition 2.1 (applied to \( \sigma = \pi \) and \( a_k = \pi(k) + k \) yields

\[
\sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} ((\pi(j) + i) - (\pi(i) + i)) = \sum_{i=1}^{n} (\pi(i) + i)(i - \pi(i)) = \sum_{i \in [n]} (i^2 - (\pi(i))^2)
\]

= \( \sum_{i \in [n]} (i^2 - (\pi(i))^2) = \sum_{i \in [n]} i^2 - \sum_{i \in [n]} (\pi(i))^2 = 0 \).
(by (1)). Hence,
\[
0 = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} ((\pi(j) + j) - (\pi(i) + i)) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} ((\pi(j) - \pi(i)) - (i - j)) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (\pi(j) - \pi(i)) - \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j).
\]

Adding \(\sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j)\) to both sides of this equality, we obtain
\[
\sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (i - j) = \sum_{1 \leq i < j \leq n; \pi(i) > \pi(j)} (\pi(j) - \pi(i)).
\]

This solves part (a) of the exercise.

(b) Let \(w_0\) denote the permutation in \(S_n\) which sends each \(k \in [n]\) to \(n + 1 - k\). Define a permutation \(\sigma \in S_n\) by \(\sigma = w_0 \circ \pi\). Thus, each \(k \in [n]\) satisfies
\[
\sigma(k) = (w_0 \circ \pi)(k) = w_0(\pi(k)) = n + 1 - \pi(k) \tag{2}
\]
(by the definition of \(w_0\)).

For any \((i, j) \in [n]^2\), we have the following chain of logical equivalences:
\[
\left(\begin{array}{c}
\sigma(i) \\
\text{(by (2))}
\end{array}\right) > \left(\begin{array}{c}
\sigma(j) \\
\text{(by (2))}
\end{array}\right) \iff (n + 1 - \pi(i) > n + 1 - \pi(j)) 
\iff (\pi(i) < \pi(j)).
\]

Thus, for any \((i, j) \in [n]^2\), the condition \((\sigma(i) > \sigma(j))\) is equivalent to \((\pi(i) < \pi(j))\). Hence, the summation sign \(\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)}\) can be rewritten as \(\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)}\). In other words, we have
\[
\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)} = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)}
\]
(an equality between summation signs). Now, part (a) of the exercise (applied to \(\sigma\) instead of \(\pi\)) yields
\[
\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j)} (\sigma(j) - \sigma(i)) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (- (j - i))
\]
\[
= - \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j - i).
\]
Comparing this with

\[
\sum_{1 \leq i < j \leq n; \sigma(i) > \sigma(j) \atop \pi(i) < \pi(j)} \sigma(j) - \sigma(i) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} ((n + 1 - \pi(j)) - (n + 1 - \pi(i))) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (- (\pi(j) - \pi(i)))
\]

we obtain

\[- \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j - i)\,.
\]

Thus,

\[
\sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (\pi(j) - \pi(i)) = \sum_{1 \leq i < j \leq n; \pi(i) < \pi(j)} (j - i)\,.
\]

This solves part (b) of the exercise.

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### 3 EXERCISE 3

#### 3.1 PROBLEM

Let \( n \) be a positive integer. For each \( p \in \mathbb{Z} \), we let

\[ D_{n,p} = \{ \sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents} \} \,.
\]

(Recall that a descent of a permutation \( \sigma \in S_n \) denotes an element \( k \in [n-1] \) satisfying \( \sigma(k) > \sigma(k + 1) \).

Let \( p \in \mathbb{Z} \). Prove that \( |D_{n,p}| = |D_{n,n-1-p}| \).

#### 3.2 SOLUTION SKETCH

We have \( n - 1 \in \mathbb{N} \) (since \( n \) is a positive integer).

Recall that if \( \sigma \in S_n \) is a permutation, then \( \text{Des} \sigma \) denotes the set of all descents of \( \sigma \).

Let \( w_0 \) denote the permutation in \( S_n \) which sends each \( k \in [n] \) to \( n + 1 - k \).
Let $\pi \in S_n$. Thus, for each $k \in [n - 1]$, we have the following chain of equivalences:

$$(k \in \text{Des}(w_0 \circ \pi)) \iff (k \text{ is a descent of } w_0 \circ \pi)$$

$$\iff \begin{pmatrix} (w_0 \circ \pi)(k) > (w_0 \circ \pi)(k + 1) \\
= w_0(\pi(k)) = n + 1 - \pi(k) \quad \text{(by the definition of } w_0) \\
= w_0(\pi(k + 1)) = n + 1 - \pi(k + 1) \quad \text{(by the definition of } w_0) 
\end{pmatrix}$$

$$\iff (n + 1 - \pi(k) > n + 1 - \pi(k + 1))$$

$$\iff (\pi(k) < \pi(k + 1)) \iff (\pi(k) \leq \pi(k + 1))$$

(since $\pi(k) = \pi(k + 1)$ can never hold (because $\pi \in S_n$))

$$\iff (\not(\pi(k) > \pi(k + 1))) \iff (k \text{ is not a descent of } \pi)$$

$$\iff (k \notin \text{Des } \pi).$$

In other words, the elements of $\text{Des}(w_0 \circ \pi)$ are precisely the elements of $[n - 1]$ that don’t belong to $\text{Des } \pi$. In other words, the set $\text{Des}(w_0 \circ \pi)$ is the complement of the set $\text{Des } \pi$ in $[n - 1]$. Thus,

$$|\text{Des}(w_0 \circ \pi)| = |[n - 1]| - |\text{Des } \pi| = n - 1 - |\text{Des } \pi|. \tag{3}$$

Now, forget that we fixed $\pi$. We thus have proven (3) for each $\pi \in S_n$.

Now, let $\pi \in D_{n,p}$. Then, $\pi$ has exactly $p$ descents $\uparrow$. In other words, $|\text{Des } \pi| = p$. Thus, (3) yields $|\text{Des}(w_0 \circ \pi)| = n - 1 - |\text{Des } \pi| = n - 1 - p$. In other words, the permutation $w_0 \circ \pi$ has exactly $n - 1 - p$ descents. In other words, $w_0 \circ \pi \in D_{n,n-1-p}$ (since the definition of $D_{n,n-1-p}$ yields $D_{n,n-1-p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } n - 1 - p \text{ descents}\}$).

Now, forget that we fixed $\pi$. We thus have proven that $w_0 \circ \pi \in D_{n,n-1-p}$ for each $\pi \in D_{n,p}$. Thus, the map

$$D_{n,p} \to D_{n,n-1-p},$$

$$\pi \mapsto w_0 \circ \pi \tag{4}$$

is well-defined. The same argument (but with $p$ replaced by $n - 1 - p$) shows that the map

$$D_{n,n-1-p} \to D_{n,n-1-(n-1-p)},$$

$$\pi \mapsto w_0 \circ \pi$$

is well-defined. In other words, the map

$$D_{n,n-1-p} \to D_{n,p},$$

$$\pi \mapsto w_0 \circ \pi \tag{5}$$

is well-defined (since $n - 1 - (n - 1 - p) = p$). But $w_0 \circ w_0 = \text{id}$ (since each $k \in [n]$ satisfies

$$(w_0 \circ w_0)(k) = w_0(w_0(k)) = n + 1 - (n + 1 - k) \quad \text{(by the definition of } w_0)$$

$$= k = \text{id}(k).$$

Thus, the two maps (4) and (5) are mutually inverse. Hence, these two maps are bijections. Thus, we have found a bijection from $D_{n,p}$ to $D_{n,n-1-p}$. Hence, $|D_{n,p}| = |D_{n,n-1-p}|$. This solves the exercise.

---

1since $\pi \in D_{n,p} = \{\sigma \in S_n \mid \sigma \text{ has exactly } p \text{ descents}\}$
3.3 Remark

1. A similar solution could have been obtained by using the permutation $\pi \circ w_0$ instead of $w_0 \circ \pi$. Indeed, similarly to (3), we also have

$$|\text{Des} (\pi \circ w_0)| = n - 1 - |\text{Des} \pi| \quad \text{for each } \pi \in S_n.$$ 

To prove this, we would have to show that

$$\text{Des} (\pi \circ w_0) = \{n - k \mid k \in [n - 1] \setminus \text{Des} \pi\}$$

(which is only a tad more complicated than proving that $\text{Des} (w_0 \circ \pi) = \{n - 1\} \setminus \text{Des} \pi$).

2. I have snuck a correction into the exercise: It used to only require $n \in \mathbb{N}$, but now it requires $n$ to be a positive integer. Indeed, the claim fails for $n = 0$. Sorry!

4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Let $S = \{s_1 < s_2 < \cdots < s_k\}$ be a subset of $[n - 1]$. Set $s_0 = 0$ and $s_{k+1} = n$. For each $i \in [k + 1]$, set $d_i = s_i - s_{i-1}$. (You might remember this construction from the definition of the map $D$ in the solution to Exercise 1 on homework set #0.)

(a) Prove that

$$|\{\sigma \in S_n \mid \text{Des} \sigma \subseteq S\}| = \binom{n}{d_1, d_2, \ldots, d_{k+1}}.$$ 

(The term on the right hand side is a multinomial coefficient. The Des $\sigma$ on the left hand side denotes the descent set of $\sigma$, that is, the set of all descents of $\sigma$.)

(b) Prove that

$$|\{\sigma \in S_n \mid \text{Des} \sigma = S\}| = \sum_{T \subseteq S} (-1)^{|S| - |T|} |\{\sigma \in S_n \mid \text{Des} \sigma \subseteq T\}|.$$ 

4.2 Solution Sketch

(a) A permutation $\sigma \in S_n$ satisfies $\text{Des} \sigma \subseteq S$ if and only if it is strictly increasing on each of the $k + 1$ intervals

$$[s_0 + 1, s_1], \quad [s_1 + 1, s_2], \quad [s_2 + 1, s_3], \quad \ldots, \quad [s_k + 1, s_{k+1}].$$ 

Hence, a permutation $\sigma \in S_n$ satisfying $\text{Des} \sigma \subseteq S$ is uniquely determined by the images

$$\sigma ([s_0 + 1, s_1]), \quad \sigma ([s_1 + 1, s_2]), \quad \sigma ([s_2 + 1, s_3]), \quad \ldots, \quad \sigma ([s_k + 1, s_{k+1}])$$

of these $k + 1$ intervals (indeed, once these images are known, we can use the strict increasingness of $\sigma$ on these intervals to reconstruct each value of $\sigma$). These images must be disjoint subsets of $[n]$ (since $\sigma$ is injective) and have the same sizes as the $k + 1$ intervals themselves (for the same reason); these sizes are

$$s_1 - s_0 = d_1, \quad s_2 - s_1 = d_2, \quad s_3 - s_2 = d_3, \quad \ldots, \quad s_{k+1} - s_k = d_{k+1}.$$ 

Thus, every permutation $\sigma \in S_n$ satisfying $\text{Des} \sigma \subseteq S$ can be constructed by the following algorithm:
• We choose a \(d_1\)-element subset of \([n]\) to be the image \(\sigma ([s_0 + 1, s_1])\). This subset can be chosen in \(\binom{n}{d_1}\) ways.

• Next, we choose a \(d_2\)-element subset of \([n]\) to be the image \(\sigma ([s_1 + 1, s_2])\), requiring that it be disjoint from the already chosen subset \(\sigma ([s_0 + 1, s_1])\). This subset can be chosen in \(\binom{n-d_1}{d_2}\) ways (because by requiring it to be disjoint from the \(d_1\)-element subset \(\sigma ([s_0 + 1, s_1])\), we are forcing it to be a \(d_2\)-element subset of the \((n-d_1)\)-element set \([n] \setminus \sigma ([s_0 + 1, s_1])\)).

• Next, we choose a \(d_3\)-element subset of \([n]\) to be the image \(\sigma ([s_2 + 1, s_3])\), requiring that it be disjoint from the already chosen subsets \(\sigma ([s_0 + 1, s_1])\) and \(\sigma ([s_1 + 1, s_2])\). This subset can be chosen in \(\binom{n-d_1-d_2}{d_3}\) ways (because by requiring it to be disjoint from the \(d_1\)-element subset \(\sigma ([s_0 + 1, s_1])\) and the \(d_2\)-element subset \(\sigma ([s_1 + 1, s_2])\), we are forcing it to be a \(d_3\)-element subset of the \((n-d_1-d_2)\)-element set \([n] \setminus \sigma ([s_0 + 1, s_1]) \setminus \sigma ([s_1 + 1, s_2])\)).

• And so on, until all \(k + 1\) images

\[
\sigma ([s_0 + 1, s_1]), \quad \sigma ([s_1 + 1, s_2]), \quad \sigma ([s_2 + 1, s_3]), \quad \ldots, \quad \sigma ([s_k + 1, s_{k+1}])
\]

are chosen. As we know, at this point, \(\sigma\) is uniquely determined.

The total number of ways in which this construction can be carried out is

\[
\binom{n}{d_1} \binom{n-d_1}{d_2} \binom{n-d_1-d_2}{d_3} \cdots \binom{n-d_1-d_2-\cdots-d_k}{d_{k+1}}
\]

\[= \prod_{i=0}^{k} \binom{n-d_1-d_2-\cdots-d_i}{d_{i+1}} = \prod_{i=1}^{k+1} \binom{n-d_1-d_2-\cdots-d_{i-1}}{d_i} = \binom{n}{d_1, d_2, \ldots, d_{k+1}}
\]

(by the first equation in Proposition 2.38 in the class notes (2018-10-03)). Thus, the number of permutations \(\sigma \in S_n\) satisfying \(\text{Des} \sigma \subseteq S\) is \(\binom{n}{d_1, d_2, \ldots, d_{k+1}}\). This solves part (a) of the exercise.

(b) We need the following result:

**Proposition 4.1.** Let \(G\) be a finite set. Let \(S\) be a subset of \(G\). Then,

\[
\sum_{I \subseteq G: \, \overline{S} \subseteq I} (-1)^{|I|} = (-1)^{|S|} [G = S].
\]

Proposition 4.1 was proven during the solution of Exercise 6 on homework set #3.

---

Of course, we are tacitly using the fact that the two already chosen subsets \(\sigma ([s_0 + 1, s_1])\) and \(\sigma ([s_1 + 1, s_2])\) are disjoint (so that the set \([n] \setminus \sigma ([s_0 + 1, s_1]) \setminus \sigma ([s_1 + 1, s_2])\) really a \((n-d_1-d_2)\)-element set).
We have
\[
\sum_{T \subseteq S} (-1)^{|S| - |T|} \left\{ \sigma \in S_n \mid \text{Des} \sigma \subseteq T \right\}
\]
\[
= \sum_{I \subseteq S} (-1)^{|S| - |I|} \left\{ \sigma \in S_n \mid \text{Des} \sigma \subseteq I \right\}
\]
\[
= \sum_{U \subseteq S} \sum_{I \subseteq S} \sum_{U \subseteq I} (-1)^{|S| - |I|} \left\{ \sigma \in S_n \mid \text{Des} \sigma \subseteq U \right\}
\]
\[
= \sum_{U \subseteq S} \sum_{I \subseteq S : U \subseteq I} (-1)^{|S| - |I|} \left\{ \sigma \in S_n \mid \text{Des} \sigma \subseteq U \right\}
\]
\[
= \sum_{U \subseteq S} \left( \sum_{I \subseteq S : U \subseteq I} (-1)^{|I|} \right) (-1)^{|S|} \left\{ \sigma \in S_n \mid \text{Des} \sigma \subseteq U \right\}
\]
\[
= \sum_{U \subseteq S} (-1)^{|U|} [S = U] (-1)^{|S|} \left\{ \sigma \in S_n \mid \text{Des} \sigma = U \right\}
\]
\[
= \sum_{U \subseteq S : U \neq S} (-1)^{|U|} [S = U] (-1)^{|S|} \left\{ \sigma \in S_n \mid \text{Des} \sigma = U \right\}
\]
\[
+ (-1)^{|S|} [S = S] (-1)^{|S|} \left\{ \sigma \in S_n \mid \text{Des} \sigma = S \right\}
\]
\[
= (-1)^{|S|} (-1)^{|S|} \left\{ \sigma \in S_n \mid \text{Des} \sigma = S \right\} = |\{ \sigma \in S_n \mid \text{Des} \sigma = S \}|.
\]

This solves part (b) of the exercise.
5 Exercise 5

5.1 Problem

Let \( n \in \mathbb{N} \). We shall follow the convention that \( t_{i,i} \) denotes the identity permutation \( \text{id} \in S_n \) for each \( i \in [n] \).

Let \( \sigma \in S_n \).

It is known that there is a unique \( n \)-tuple \( (i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n] \) satisfying \( \sigma = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n} \). (See [Grinbe16, Exercise 5.9] for the proof of this fact, or – easier – do it on your own.) Consider this \( n \)-tuple. (It is sometimes called the transposition code of \( \sigma \).)

For each \( k \in \{0, 1, \ldots, n\} \), we define a permutation \( \sigma_k \in S_n \) by \( \sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} \).

Note that this permutation \( \sigma_k \) leaves each of the numbers \( k + 1, k + 2, \ldots, n \) unchanged (since all of \( i_1, i_2, \ldots, i_k \), as well as \( 1, 2, \ldots, k \), are \( \leq k \)).

For each \( k \in [n] \), let \( m_k = \sigma_k (k) \).

(a) Show that \( m_k \in [k] \) for all \( k \in [n] \).

(b) Show that \( \sigma_k (i_k) = k \) for all \( k \in [n] \).

(c) Show that \( \sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n} \).

(d) Let \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) be any \( 2n \) numbers. Prove that

\[
\sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} x_k y_{\sigma(k)} = \sum_{k=1}^{n} (x_{i_k} - x_k) (y_{m_k} - y_k).
\]

(e) Now assume that the numbers \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) are real and satisfy \( x_1 \geq x_2 \geq \cdots \geq x_n \) and \( y_1 \geq y_2 \geq \cdots \geq y_n \). Conclude that

\[
\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\sigma(k)}.
\]

5.2 Remark

Parts (a) and (c), combined, show that \( (m_1, m_2, \ldots, m_n) \) is the transposition code of \( \sigma^{-1} \).

Part (e) of the exercise is known as the rearrangement inequality. The proof in this exercise is far from its easiest proof, but has the advantage of “manifest positivity” – i.e., it gives an explicit formula for the difference between the two sides as a sum of products of nonnegative numbers.

5.3 Solution sketch

Let us first notice that any two elements \( u, v \in [n] \) and any permutation \( \pi \in S_n \) satisfy

\[
t_{\pi(u), \pi(v)} \circ \pi = \pi \circ t_{u,v}.
\]

[Proof of (6): Let \( u, v \in [n] \) and \( \pi \in S_n \). Fix \( k \in [n] \). We shall prove that \( (t_{\pi(u), \pi(v)} \circ \pi) (k) = (\pi \circ t_{u,v}) (k) \).

Indeed, we are in one of the following three cases:

Case 1: We have \( k = u \).

Case 2: We have \( k = v \).

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Case 3: We have neither \( k = u \) nor \( k = v \).

Let us first consider Case 1. In this case, we have \( k = u \). Thus, \( t_{u,v}(k) = t_{u,v}(u) = v \) (independently of whether \( u = v \) or \( u \neq v \)). Also, from \( k = u \), we obtain
\[
(t_{\pi(u),\pi(v)} \circ \pi)(k) = (t_{\pi(u),\pi(v)} \circ \pi)(u) = t_{\pi(u),\pi(v)}(\pi(u)) = \pi(v)
\]
(again, independently of whether \( \pi(u) = \pi(v) \) holds or not). Comparing this with
\[
(\pi \circ t_{u,v})(k) = \pi(t_{u,v}(k)) = \pi(v) \quad \text{(since } t_{u,v}(k) = v),
\]
we obtain \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \). Hence, \((t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k)\) is proven in Case 1.

The argument in Case 2 is analogous, and we leave it to the reader.

Let us now consider Case 3. In this case, we have neither \( k = u \) nor \( k = v \). Thus, \( t_{u,v}(k) = k \) (independently of whether \( u = v \) or \( u \neq v \)). Also, recall that we have neither \( k = u \) nor \( k = v \). Thus, we have neither \( \pi(k) = \pi(u) \) nor \( \pi(k) = \pi(v) \) (since the map \( \pi \) is injective (because \( \pi \in S_n \)). Hence, \( t_{\pi(u),\pi(v)}(\pi(k)) = \pi(k) \) (again, independently of whether \( \pi(u) = \pi(v) \) holds or not). Now,
\[
(t_{\pi(u),\pi(v)} \circ \pi)(k) = t_{\pi(u),\pi(v)}(\pi(k)) = \pi(k).
\]
Comparing this with
\[
(\pi \circ t_{u,v})(k) = \pi(t_{u,v}(k)) = \pi(k) \quad \text{(since } t_{u,v}(k) = k),
\]
we obtain \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \). Hence, \((t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k)\) is proven in Case 3.

We have now proven \( (t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k) \) in each of the three Cases 1, 2 and 3. Thus, \((t_{\pi(u),\pi(v)} \circ \pi)(k) = (\pi \circ t_{u,v})(k)\) always holds.

Forget now that we fixed \( k \). We thus have shown that \( t_{\pi(u),\pi(v)}(\pi(k)) = (\pi \circ t_{u,v})(k) \) for each \( k \in [n] \). In other words, \( t_{\pi(u),\pi(v)} \circ \pi = \pi \circ t_{u,v} \). Thus, \( [\text{9}] \) is proven.

Recall that \((i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]\). Thus,
\[
i_j \in [j] \quad \text{for each } j \in [n].
\]

The definition of \( \sigma_0 \) shows that
\[
\sigma_0 = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{0,i_0} = (\text{composition of 0 permutations}) = \text{id}.
\]
The definition of \( \sigma_n \) shows that
\[
\sigma_n = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n} = \sigma.
\]

(a) Let \( k \in [n] \). Then, from [7], we conclude that each \( j \in [k] \) satisfies \( i_j \in [j] \subseteq [k] \) (since \( j \leq k \)). Hence, the \( k \) numbers \( i_1, i_2, \ldots, i_k \) all belong to \([k]\). The same holds for the \( k \) numbers \( 1, 2, \ldots, k \). Thus, the \( k \) permutations \( t_{1,i_1}, t_{2,i_2}, \ldots, t_{k,i_k} \) all preserve the set \([k]\) \text{[8]}. Hence, their composition \( t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} \) preserves the set \([k]\) as well. In view of

---

\(^3\)We say that a map \( \tau : [n] \rightarrow [n] \) preserves a subset \( S \) of \([n]\) if and only if it satisfies \( \tau(S) \subseteq S \). This does not mean that \( \tau(s) = s \) for each \( s \in S \); it only means that \( \tau \) sends each element of \( S \) to a (possibly different) element of \( S \).

\(^4\)Here, we are using the following fact: If \( S \) is a subset of \([n]\), and if \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are \( k \) maps from \([n]\) to \([n]\) that all preserve the set \( S \), then the composition \( \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k \) of these \( k \) maps must preserve the set \( S \) as well. (This is easy to prove by induction on \( k \).)
\( \sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} \), this rewrites as follows: The map \( \sigma_k \) preserves the set \([k]\). In other words, \( \sigma_k ([k]) \subseteq [k] \). Now, \( k \in [k] \), so that \( \sigma_k (k) \in \sigma_k ([k]) \subseteq [k] \). Hence, \( m_k = \sigma_k (k) \in [k] \).

This solves part (a) of the exercise.

(b) Let \( k \in [n] \). Then, from (7), we conclude that each \( j \in [k - 1] \) satisfies \( i_j \in [j] \subseteq [k - 1] \) (since \( j \leq k - 1 \)). Hence, the \( k - 1 \) numbers \( i_1, i_2, \ldots, i_{k-1} \) all belong to \([k - 1]\). The same holds for the \( k - 1 \) numbers \( 1, 2, \ldots, k - 1 \). Thus, the \( k - 1 \) permutations \( t_{1,i_1}, t_{2,i_2}, \ldots, t_{k-1,i_{k-1}} \) all leave each of the numbers \( k, k + 1, \ldots, n \) unchanged. Hence, their composition \( t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}} \) leaves each of the numbers \( k, k + 1, \ldots, n \) unchanged. In particular, it thus leaves the number \( k \) unchanged. In other words,

\[
(t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) (k) = k.
\]

The definition of \( \sigma_k \) yields

\[
\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ t_{k,i_k}.
\]

Hence,

\[
\sigma_k (i_k) = ((t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ t_{k,i_k}) (i_k) = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) (t_{k,i_k} (i_k)) = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) (k) = k.
\]

This solves part (b) of the exercise.

(c) We shall show that

\[
\sigma_p^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{p,m_p} \quad \text{for each } p \in \{0, 1, \ldots, n\}. \quad (8)
\]

[Proof of (8): We shall prove (8) by induction on \( p \):

Induction base: In the case of \( p = 0 \), the equality (8) holds, since \( \sigma_0 \) is defined as an empty composition whereas the right hand side of (8) also is an empty composition in this case. This completes the induction base.

Induction step: Let \( k \in [n] \). Assume that (8) holds for \( p = k - 1 \). We must prove that (8) holds for \( p = k \).

We have assumed that (8) holds for \( p = k - 1 \). That is, we have

\[
\sigma_{k-1}^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{k-1,m_{k-1}}.
\]

Part (b) of the exercise yields \( \sigma_k (i_k) = k \), whereas the definition of \( m_k \) yields \( \sigma_k (k) = m_k \). But (8) (applied to \( \pi = \sigma_k \), \( u = i_k \) and \( v = k \)) yields

\[
t_{\sigma_k (i_k), \sigma_k (k)} \circ \sigma_k = \sigma_k \circ t_{i_k,k}.
\]

In view of \( \sigma_k (i_k) = k \) and \( \sigma_k (k) = m_k \), this rewrites as

\[
t_{k,m_k} \circ \sigma_k = \sigma_k \circ t_{i_k,k} = \sigma_k \circ t_{k,i_k}.
\]

We have \( \sigma_{k-1} = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}} \) (by the definition of \( \sigma_{k-1} \)). Now, the definition of \( \sigma_k \) yields

\[
\sigma_k = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k,i_k} = (t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{k-1,i_{k-1}}) \circ t_{k,i_k} = \sigma_{k-1} \circ t_{k,i_k}.
\]

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Solving this equation for \(\sigma_{k-1}\), we obtain
\[
\sigma_{k-1} = \sigma_k \circ t_{k,i_k}^{-1} = \sigma_k \circ t_{k,m_k} = t_{k,m_k} \circ \sigma_k \quad \text{ (by \([9]\)).}
\]

Solving this equation for \(\sigma_k\), we find
\[
\sigma_k = \left( t_{k,m_k} \circ \sigma_k \right)^{-1} = \left( t_{k,m_k} \circ \sigma_k \right)^{-1} \circ t_{k,m_k} = t_{k,m_k} \circ \sigma_k \quad \text{ (by \([8]\)).}
\]
Hence,
\[
\sigma_k^{-1} = (t_{k,m_k} \circ \sigma_k)^{-1} = \left( t_{k,m_k} \circ \sigma_k^{-1} \circ t_{k,m_k}^{-1} \right)^{-1} = (t_{k,m_k} \circ \sigma_k)^{-1} = \left( t_{k,m_k} \circ \sigma_k^{-1} \circ t_{k,m_k}^{-1} \right)^{-1} = \left( t_{k,m_k} \circ \sigma_k \circ t_{k,m_k}^{-1} \right) \circ \sigma_k^{-1} = \sigma_{k-1} \circ \sigma_{k-1}.
\]
In other words, \([12]\) holds for \(p = k\). This completes the induction step. Thus, \([8]\) is proven by induction.

Applying \([8]\) to \(p = n\), we obtain \(\sigma_n^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}\). In view of \(\sigma_n = \sigma\), this rewrites as \(\sigma^{-1} = t_{1,m_1} \circ t_{2,m_2} \circ \cdots \circ t_{n,m_n}\). This solves part (c) of the exercise.

(d) For each permutation \(\tau \in S_n\), we define a number \(z(\tau)\) by
\[
z(\tau) = \sum_{k=1}^{n} x_k y_{\tau(k)}.
\]
We shall show that
\[
z(\sigma_{p-1}) - z(\sigma_{p}) = (x_{i_p} - x_p) (y_{m_p} - y_p) \quad \text{ for each } p \in [n].
\]

[Proof of \([12]\): Let \(p \in [n]\). Applying \([10]\) to \(k = p\), we obtain \(\sigma_p = \sigma_{p-1} \circ t_{p,i_p}\). Hence, if \(p = i_p\), then \([12]\) holds. Thus, for the rest of this proof, we WLOG assume that \(p \neq i_p\). Hence, \(t_{p,i_p}\) is an actual transposition (not the identity map).
From \(\sigma_p = \sigma_{p-1} \circ t_{p,i_p}\), we obtain
\[
\sigma_p(p) = \left( \sigma_{p-1} \circ t_{p,i_p} \right)(p) = \sigma_{p-1} \left( t_{p,i_p}(p) \right) = \sigma_{p-1}(i_p),
\]
so that
\[
\sigma_{p-1}(i_p) = \sigma_p(p) = m_p \quad \text{ (since the definition of } m_p \text{ yields } m_p = \sigma_p(p)\).
\]
From \(\sigma_p = \sigma_{p-1} \circ t_{p,i_p}\), we also obtain
\[
\sigma_p(i_p) = \left( \sigma_{p-1} \circ t_{p,i_p} \right)(i_p) = \sigma_{p-1} \left( t_{p,i_p}(i_p) \right) = \sigma_{p-1}(p),
\]
\(\text{ since } t_{p,i_p} = \text{id}\). Thus, the left hand side of \([12]\) equals 0 as well. Hence, the equality \([12]\) holds (since both its right hand side and its left hand side equal 0).

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so that
\[ \sigma_{p-1}(p) = \sigma_p(i_p) = p \] (14)

(by part (b) of the exercise, applied to \( k = p \)).

Every \( k \in [n] \) satisfying \( k \neq p \) and \( k \neq i_p \) satisfies
\[ \sigma_{p-1}(k) = \sigma_p(k) \] (15)

Now, the definition of \( z(\sigma_{p-1}) \) yields
\[
z(\sigma_{p-1}) = \sum_{k=1}^{n} x_k y_{\sigma_{p-1}(k)} = \underbrace{x_p y_{\sigma_{p-1}(p)}}_{=y_{\sigma_p(p)} = y_{m_p}} + \underbrace{x_{i_p} y_{\sigma_{p-1}(i_p)}}_{=y_{\sigma_p(i_p)} = y_p} + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_{p-1}(k)} \]
\[= x_p y_{\sigma_p(p)} + x_{i_p} y_{\sigma_p(i_p)} + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_p(k)} \]
\[
\quad \text{(here, we have split the addends for } k = p \text{ and for } k = i_p \text{ from the sum (and these are two distinct addends, since } p \neq i_p) \]
\[= x_p y_{\sigma_p(p)} + x_{i_p} y_{\sigma_p(i_p)} + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_p(k)} \]

On the other hand, the definition of \( z(\sigma_p) \) yields
\[
z(\sigma_p) = \sum_{k=1}^{n} x_k y_{\sigma_p(k)} = \underbrace{x_p y_{\sigma_p(p)}}_{\text{since } \sigma_p(p) = m_p} + \underbrace{x_{i_p} y_{\sigma_p(i_p)}}_{\text{since } \sigma_p(i_p) = p} + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_p(k)} \]
\[= x_p y_{m_p} + x_{i_p} y_p + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_p(k)} \]

Subtracting this equality from the preceding equality, we obtain
\[
z(\sigma_{p-1}) - z(\sigma_p) = \left( x_p y_{\sigma_p(p)} + x_{i_p} y_{\sigma_p(i_p)} + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_p(k)} \right) - \left( x_p y_{m_p} + x_{i_p} y_p + \sum_{k\in[n]; \ k\neq p \text{ and } k\neq i_p} x_k y_{\sigma_p(k)} \right)
\]
\[= x_p y_{\sigma_p(p)} - x_p y_{m_p} + (x_{i_p} - x_p)(y_p - y_{\sigma_p(p)}) \]
\[
= x_p y_{\sigma_p(p)} - x_p y_{m_p} - x_{i_p} y_p + x_{i_p} y_{m_p} - x_{i_p} y_p = (x_{i_p} - x_p)(y_{m_p} - y_p) .
\]

This proves (12).

*Proof: Let \( k \in [n] \) be such that \( k \neq p \) and \( k \neq i_p \). Thus, \( t_{p,i_p}(k) = k \). But \( \sigma_p = \sigma_{p-1} \circ t_{p,i_p} \); hence,
\[ \sigma_p(k) = (\sigma_{p-1} \circ t_{p,i_p})(k) = \sigma_{p-1}(t_{p,i_p}(k)) = \sigma_{p-1}(k) \]
so that \( \sigma_{p-1}(k) = \sigma_p(k) \), qed.\]
Now, the telescope principle yields
\[
\sum_{p=1}^{n} (z(\sigma_{p-1}) - z(\sigma_p)) = z\begin{pmatrix} \sigma_0 \\ \equiv \text{id} \end{pmatrix} - z\begin{pmatrix} \sigma_p \\ \equiv \sigma \end{pmatrix} = z(\text{id}) - z(\sigma) = \sum_{k=1}^{n} x_k y_{\sigma(k)} - \sum_{k=1}^{n} x_k y_{\sigma(k)}. 
\]
(by the definition of \(z(\sigma)\))

Hence,
\[
\sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} x_k y_{\sigma(k)} = \sum_{p=1}^{n} (z(\sigma_{p-1}) - z(\sigma_p)) = \sum_{p=1}^{n} (x_{i_p} - x_p) (y_{m_p} - y_p) = \sum_{k=1}^{n} (x_{i_k} - x_k) (y_{m_k} - y_k)
\]
(by (12))

(here, we have renamed the summation index \(p\) as \(k\)). This solves part (d) of the exercise.

(e) Fix \(k \in [n]\). Then, \(i_k \in [k]\) (by (7)), so that \(i_k \leq k\) and therefore \(x_{i_k} \geq x_k\) (since \(x_1 \geq x_2 \geq \cdots \geq x_n\)). Hence, \(x_{i_k} - x_k \geq 0\).

Also, \(m_k \in [k]\) (by part (a) of the exercise), so that \(m_k \leq k\) and thus \(y_{m_k} \geq y_k\) (since \(y_1 \geq y_2 \geq \cdots \geq y_n\)). Hence, \(y_{m_k} - y_k \geq 0\). Now,
\[
(x_{i_k} - x_k) (y_{m_k} - y_k) \geq 0.
\]
(by (16))

(16)

Now, forget that we fixed \(k\). We thus have proven (16) for each \(k \in [n]\). Now, part (d) of the exercise yields
\[
\sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} x_k y_{\sigma(k)} = \sum_{k=1}^{n} (x_{i_k} - x_k) (y_{m_k} - y_k) \geq 0.
\]
(by (16))

In other words,
\[
\sum_{k=1}^{n} x_k y_k \geq \sum_{k=1}^{n} x_k y_{\sigma(k)}.
\]
This solves part (e) of the exercise.

6 Exercise 6

6.1 Problem

Prove the following:
(a) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are such that $m < n$, then
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0. \]

(b) If $n \in \mathbb{N}$ and $r \in [n-1]$, then
\[ \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0. \]

6.2 Solution sketch

(a) First solution to part (a): Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $m < n$. We have $|[m]| = m < n = |[n]|$. Thus, there are no surjections from $[m]$ to $[n]$ (by the Pigeonhole Principle for Surjections). Recall that sur $(m, n)$ denotes the number of all surjections from $[m]$ to $[n]$. Thus, sur $(m, n) = 0$ (since there are no surjections from $[m]$ to $[n]$).

But Theorem 2.28 from [class (2018-10-01)] shows that
\[ \text{sur} (m, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m \]

(here, we have renamed the summation index $i$ as $k$). Comparing this with sur $(m, n) = 0$, we obtain \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = 0. \] This solves part (a) of the exercise.

Second solution to part (a): Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $m < n$. Exercise 6 (b) on homework set #3 yields that if $A_1, A_2, \ldots, A_n$ are $n$ numbers, then
\[ \sum_{I \subseteq [n]} (-1)^{|I|} \left( \sum_{i \in I} A_i \right) \right)^m = 0. \]

Applying this to $A_i = 1$, we obtain
\[ \sum_{I \subseteq [n]} (-1)^{|I|} \left( \sum_{i \in I} 1 \right) \right)^m = 0. \]
Thus,

\[
0 = \sum_{I \subseteq [n]} (-1)^{n-|I|} \left( \sum_{1 \leq i \leq |I|} 1 \right)^m = \sum_{I \subseteq [n]} (-1)^{n-|I|} |I|^m
\]

\[
= \sum_{i=0}^{n} \sum_{\substack{I \subseteq [n]; |I|=i \atop i=0}} (-1)^{n-|I|} i^m = \sum_{i=0}^{n} (-1)^{n-i} i^m
\]

\[
= \sum_{i=0}^{n} \text{(the number of all } I \subseteq [n] \text{ satisfying } |I|=i \text{)} \cdot (-1)^{n-i} i^m
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^{n-i} i^m
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^{n-i} i^m
\]

\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.
\]

This solves part (a) of the exercise again.

*Third solution to part (a):* Part (a) of the exercise is a particular case of Theorem 6.1 further below (applied to \(a = m\), \(b = n\) and \(c = n\)).

(b) We need a generalization of part (a) of the exercise:

**Theorem 6.1.** Let \(a \in \mathbb{N}\), \(b \in \mathbb{Q}\) and \(c \in \mathbb{N}\) be such that \(c > a\). Then,

\[
\sum_{k=0}^{c} (-1)^k \binom{c}{k} (b-k)^a = 0.
\]

For the proof of Theorem 6.1 see [Grinbe18, Theorem 0.2].

Let \(n \in \mathbb{N}\) and \(r \in \{n-1\}\). We have \(r \leq n-1\), thus \(r \leq n-1\) and therefore \(2r \leq 2(n-1) < 2n\). Thus, \(2n > 2r\). Hence, Theorem 6.1 (applied to \(a = 2r\), \(b = n\) and \(c = 2n\)) yields

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 0.
\]
Thus,
\[
0 = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}
\]
\[
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}
\]

(since 0 ≤ n ≤ 2n). But
\[
\sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}
\]
\[
= \sum_{k=0}^{n-1} (-1)^{2n-k} \binom{2n}{2n-k} \binom{n-(2n-k)}{k}^{2r}
\]
\[
= \binom{2n}{k}
\]
(by the symmetry of Pascal’s triangle)

(here, we have substituted 2n – k for k in the sum)
\[
= \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (- (n-k))^{2r} = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (n-k)^{2r}
\]

(since 2r is even)
\[
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} - (-1)^n \binom{2n}{n} (n-n)^{2r}
\]

(here, we have extended the range of the sum to include a new addend for k = n, and then subtracted that addend)
\[
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}.
\]

Hence, (17) becomes
\[
0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \sum_{k=n+1}^{2n} (-1)^k \binom{2n}{k} (n-k)^{2r}
\]
\[
= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} + \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r} = 2 \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}.
\]

Dividing this equality by 2, we find 0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (n-k)^{2r}. This solves part (b) of the exercise.

6.3 Remark

I have learnt part (b) of the exercise from MathOverflow question #312839, which also asks if the sum is ≠ 0 when 2r is replaced by an odd integer between 1 and 2n – 1.
7 EXERCISE 7

7.1 PROBLEM

Let \( n \in \mathbb{N} \) and \( d \in \mathbb{N} \). An \( n \)-tuple \((x_1, x_2, \ldots, x_n) \in [d]^n\) is said to be **all-even** if each element of \([d]\) occurs an even number of times in this \( n \)-tuple (i.e., if for each \( k \in [d] \), the number of all \( i \in [n] \) satisfying \( x_i = k \) is even). For example, the 4-tuple \((1, 4, 4, 1)\) and the 6-tuples \((1, 3, 3, 5, 1, 5)\) and \((2, 4, 2, 4, 3, 3)\) are all-even, while the 4-tuples \((1, 2, 2, 4)\) and \((2, 4, 6, 4)\) are not.

Prove that the number of all all-even \( n \)-tuples \((x_1, x_2, \ldots, x_n) \in [d]^n\) is

\[
\frac{1}{2^d} \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

**Hint:** Compute the sum \( \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n \) in two ways. One way is to split it according to the number of \( i \in [d] \) satisfying \( e_i = -1 \); this is a number \( k \in \{0, 1, \ldots, d\} \). Another way is by using the product rule:

\[
(e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1}e_{x_2}\cdots e_{x_n}
\]

and then simplifying each sum \( \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1}e_{x_2}\cdots e_{x_n} \) using a form of destructive interference. This is not unlike the number of 1-even \( n \)-tuples, which we computed at the end of the 2018-10-10 class.

7.2 SOLUTION SKETCH

Recall the **product rule** (which we have already used when solving Exercise 6 on homework set #3):

**Proposition 7.1** (Product rule). Let \( m \in \mathbb{N} \). Let \( I \) be a finite set. Let \( P_{u,v} \), for all \( u \in [m] \) and \( v \in I \), be numbers or polynomials or square matrices of the same size. Then,

\[
\left( \sum_{i \in I} P_{1,i} \right) \left( \sum_{i \in I} P_{2,i} \right) \cdots \left( \sum_{i \in I} P_{m,i} \right) = \sum_{(i_1, i_2, \ldots, i_m) \in I^m} P_{1,i_1}P_{2,i_2}\cdots P_{m,i_m}.
\]

Fix a \( d \)-tuple \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d\). We now apply Proposition 7.1 to \( m = n \), \( I = [d] \) and \( P_{u,v} = e_v \). As a result, we obtain

\[
\left( \sum_{i \in [d]} e_i \right) \left( \sum_{i \in [d]} e_i \right) \cdots \left( \sum_{i \in [d]} e_i \right) \quad \text{\( n \) times} = \sum_{(i_1, i_2, \ldots, i_n) \in [d]^n} e_{i_1}e_{i_2}\cdots e_{i_n} = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1}e_{x_2}\cdots e_{x_n}
\]
(here, we have renamed the summation index \((i_1, i_2, \ldots, i_n)\) as \((x_1, x_2, \ldots, x_n)\)). Thus,

\[
\sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n} = \left( \sum_{i \in [d]} e_i \right) \left( \sum_{i \in [d]} e_i \right) \cdots \left( \sum_{i \in [d]} e_i \right) = \left( \sum_{i \in [d]} e_i \right)^n = (e_1 + e_2 + \cdots + e_d)^n.
\]  

(18)

Now, forget that we fixed \((e_1, e_2, \ldots, e_d)\). We thus have proven the equality (18) for each \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d\).

Now,

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n}
\]

(by (18))

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} e_{x_1} e_{x_2} \cdots e_{x_n} = \sum_{(x_1, x_2, \ldots, x_n) \in [d]^n} \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}.
\]  

(19)

We shall now simplify the inner sum on the right hand side of this equality. Indeed, we claim the following:

**Claim 1:** Let \((x_1, x_2, \ldots, x_n) \in [d]^n\).

\(\textbf{(a)}\) If the \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is not all-even, then

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = 0.
\]

\(\textbf{(b)}\) If the \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is all-even, then

\[
\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = 2^d.
\]

**Proof of Claim 1:** (a) Assume that the \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is not all-even. Thus, it is not true that for each \(k \in [d]\), the number of all \(i \in [n]\) satisfying \(x_i = k\) is even (by the definition of “all-even”). In other words, there exists some \(k \in [d]\) such that the number of all \(i \in [n]\) satisfying \(x_i = k\) is odd. Consider this \(k\).

The number

\[
\sum_{i \in [n]} [x_i = k] = \sum_{i \in [n]: x_i = k} + \sum_{i \in [n]: x_i \neq k} \text{ (since } x_i = k \text{)} + \sum_{i \in [n]: x_i \neq k} \text{ (since } x_i \neq k) = \sum_{i \in [n]: x_i = k} + \sum_{i \in [n]: x_i \neq k} + \sum_{i \in [n]: x_i \neq k}
\]

\(= \sum_{i \in [n]: x_i = k} = 1\) (the number of all \(i \in [n]\) satisfying \(x_i = k\)) \cdot 1

\(= (\text{the number of all } i \in [n] \text{ satisfying } x_i = k)\)
is odd (by the definition of \( k \)). Now,

\[
(-1)^{[x_1=k]} (-1)^{[x_2=k]} \cdots (-1)^{[x_n=k]} = \prod_{i \in [n]} (-1)^{[x_i=k]} = (-1)^{\sum_{i \in [n]} [x_i=k]} = -1
\]

(since the number \( \sum_{i \in [n]} [x_i = k] \) is odd).

Now, define the two subsets

\[
N = \{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d \mid e_k = -1\}
\]

and

\[
P = \{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d \mid e_k = 1\}
\]

of the set \( \{-1, 1\}^d \). Clearly, each element of \( \{-1, 1\}^d \) belongs to exactly one of these two subsets \( N \) and \( P \) (because for each \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d\), we have either \( e_k = -1 \) or \( e_k = 1 \) but not both).

Clearly, the map

\[
N \to P, \quad (e_1, e_2, \ldots, e_d) \mapsto (e_1, e_2, \ldots, e_{k-1}, -e_k, e_{k+1}, e_{k+2}, \ldots, e_d)
\]

(which replaces the \( k \)-th entry of a \( d \)-tuple by its negative, while leaving all other entries unchanged) is well-defined and bijective (indeed, its inverse map is defined by the same rule). We can rewrite this map (using the Iverson bracket notation) as the map

\[
N \to P, \quad (e_1, e_2, \ldots, e_d) \mapsto \left(\left(-1\right)^{[1=k]} e_1, \left(-1\right)^{[2=k]} e_2, \ldots, \left(-1\right)^{[d=k]} e_d\right)
\]

(because each \((e_1, e_2, \ldots, e_d) \in N\) satisfies

\[
\left(\left(-1\right)^{[1=k]} e_1, \left(-1\right)^{[2=k]} e_2, \ldots, \left(-1\right)^{[d=k]} e_d\right) = (e_1, e_2, \ldots, e_{k-1}, -e_k, e_{k+1}, e_{k+2}, \ldots, e_d)
\]

\footnote{Proof. Let \((e_1, e_2, \ldots, e_d) \in N\). Then, each \( i \in [d] \) satisfying \( i \neq k \) satisfies \([i = k] = 0\) and therefore

\[
\left(-1\right)^{[i=k]} e_i = \sum_{\substack{i \in [d] \mid i = k}} \left(-1\right)^{[i=k]} e_i = e_i.
\]

Hence, the \( d \)-tuple \(\left(\left(-1\right)^{[1=k]} e_1, \left(-1\right)^{[2=k]} e_2, \ldots, \left(-1\right)^{[d=k]} e_d\right)\) differs from

the \( d \)-tuple \((e_1, e_2, \ldots, e_d)\) only in its \( k \)-th entry. As for its \( k \)-th entry, it is \([k=k] e_k = -e_k\). Thus,

\[
\left(-1\right)^{[1=k]} e_1, \left(-1\right)^{[2=k]} e_2, \ldots, \left(-1\right)^{[d=k]} e_d\]
\]

is obtained from the \( d \)-tuple \((e_1, e_2, \ldots, e_d)\) by replacing its \( k \)-th entry by \(-e_k\). In other words,

\[
\left(-1\right)^{[1=k]} e_1, \left(-1\right)^{[2=k]} e_2, \ldots, \left(-1\right)^{[d=k]} e_d = (e_1, e_2, \ldots, e_{k-1}, -e_k, e_{k+1}, e_{k+2}, \ldots, e_d).
\]}

is bijective, i.e., is a bijection from \( N \) to \( P \).
Recall that each element of $\{-1, 1\}^d$ belongs to exactly one of the two subsets $N$ and $P$. Hence, we can split the sum $\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n}$ as follows:

$$\sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} e_{x_1} e_{x_2} \cdots e_{x_n} = \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} + \sum_{(e_1, e_2, \ldots, e_d) \in P} e_{x_1} e_{x_2} \cdots e_{x_n}$$

$$= \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} + \sum_{(e_1, e_2, \ldots, e_d) \in P} \left( (-1)^{[x_1=k]} e_{x_1} \right) \left( (-1)^{[x_2=k]} e_{x_2} \right) \cdots \left( (-1)^{[x_n=k]} e_{x_n} \right)$$

$$= \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} + \sum_{(e_1, e_2, \ldots, e_d) \in N} \left( (-1)^{[x_1=k]} (-1)^{[x_2=k]} \cdots (-1)^{[x_n=k]} \right) (e_{x_1} e_{x_2} \cdots e_{x_n})$$

$$= \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} - \sum_{(e_1, e_2, \ldots, e_d) \in N} e_{x_1} e_{x_2} \cdots e_{x_n} = 0.$$

This proves Claim 1 (a).

(b) Assume that the $n$-tuple $(x_1, x_2, \ldots, x_n)$ is all-even. Thus, for each $k \in [d]$, the number of all $i \in [n]$ satisfying $x_i = k$ is even (by the definition of “all-even”).

Let $k \in [d]$. As we have just seen, the number of all $i \in [n]$ satisfying $x_i = k$ is even. In other words, there exists some $h \in \mathbb{Z}$ such that

$$\text{(the number of all } i \in [n] \text{ satisfying } x_i = k) = 2h. \quad (20)$$

Consider this $h$.

Now, let $(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d$ be arbitrary. Thus, $e_k \in \{-1, 1\}$, so that $e_k^2 = 1$.

Now,

$$\prod_{i \in [n]; \ x_i=k} e_{x_i} = \prod_{i \in [n]; \ x_i=k} e^2 = e_k^{\text{the number of all } i \in [n] \text{ satisfying } x_i = k} = e_k^{2h} \quad \text{(by (20))}$$

$$= \left( e_k^2 \right)^h = 1^h = 1. \quad (21)$$

Now, forget that we fixed $(e_1, e_2, \ldots, e_d)$ and $k$. We thus have proven (21) for each $(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d$ and $k \in [d]$.

Now, each $(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d$ satisfies

$$e_{x_1} e_{x_2} \cdots e_{x_n} = \prod_{i \in [n]} e_{x_i} = \prod_{k \in [d]} \prod_{i \in [n]; \ x_i=k} e_{x_i} = \prod_{k \in [d]} 1 = 1.$$
Hence,

\[
\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} e_1 e_2 \cdots e_x = \sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} 1 = \left|\{-1,1\}^d\right| \cdot 1 = \left|\{-1,1\}^d\right| = |\{-1,1\}|^d = 2^d.
\]

This proves Claim 1 (b).]

Now, (19) becomes

\[
\sum_{(e_1,e_2,\ldots,e_d)\in\{-1,1\}^d} (e_1 + e_2 + \cdots + e_d)^n = \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n} e_{x_1} e_{x_2} \cdots e_{x_n} = \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n; (x_1,x_2,\ldots,x_n) \text{ is all-even}} (e_1,e_2,\ldots,e_d)\in\{-1,1\}^d 
\]

\[
= 2^d + \sum_{(x_1,x_2,\ldots,x_n)\in[d]^n; (x_1,x_2,\ldots,x_n) \text{ is not all-even}} (e_1,e_2,\ldots,e_d)\in\{-1,1\}^d 
\]

\[
= (\text{the number of all all-even } (x_1,x_2,\ldots,x_n) \in [d]^n) \cdot 2^d. \quad (22)
\]

For each \(d\)-tuple \((e_1,e_2,\ldots,e_d) \in \{-1,1\}^d\), we have

\[
d - (e_1 + e_2 + \cdots + e_d) = \frac{d}{2} \sum_{i=1}^{d/2} e_i = \sum_{i\in[d]} 1 - \sum_{i\in[d]} e_i = \sum_{i\in[d]} (1 - e_i)
\]

\[
= \sum_{i\in[d]; e_i=-1} \left(1 - e_i\right) + \sum_{i\in[d]; e_i=1} \left(1 - e_i\right) 
\]

(since each \(i \in [d]\) satisfies either \(e_i = -1\) or \(e_i = 1\) (but not both) (because \((e_1,e_2,\ldots,e_d) \in \{-1,1\}^d\))

\[
= \sum_{i\in[d]; e_i=-1} (1 - (1)) + \sum_{i\in[d]; e_i=1} (1 - 1) = \sum_{i\in[d]; e_i=-1} 2 + \sum_{i\in[d]; e_i=1} 0 
\]

\[
= \sum_{i\in[d]; e_i=-1} 2 = |\{i \in [d] \mid e_i = -1\}| \cdot 2 = 2 \cdot |\{i \in [d] \mid e_i = -1\}|
\]

and thus

\[
e_1 + e_2 + \cdots + e_d = d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|
\]
Comparing this with (22), we obtain

\[ \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n = \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n. \]  

(23)

Hence, (23) becomes

\[ \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n. \]

On the other hand, a \(d\)-tuple \((e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d\) is uniquely determined by the set \(\{i \in [d] \mid e_i = -1\}\) of all positions at which it contains a \(-1\) (and conversely, for every subset \(S\) of \([d]\), there exists such a \(d\)-tuple whose set \(\{i \in [d] \mid e_i = -1\}\) is \(S\)). Thus, the map

\[ \{-1, 1\}^d \to \{S \subseteq [d]\}, \quad (e_1, e_2, \ldots, e_d) \mapsto \{i \in [d] \mid e_i = -1\} \]

is a bijection. Hence, we can substitute \(S\) for \(\{i \in [d] \mid e_i = -1\}\) in the sum on the right hand side of (23). We thus obtain

\[ \sum_{(e_1, e_2, \ldots, e_d) \in (-1, 1)^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n \]

\[ = \sum_{S \subseteq [d]} (d - 2 \cdot |S|)^n = \sum_{k=0}^{d} \sum_{S \subseteq [d]; |S|=k} \left( d - 2 \cdot \frac{|S|}{k} \right)^n \]

\[ = \sum_{k=0}^{d} \left( d - 2k \right)^n \]

\[ = \sum_{k=0}^{d} \left( \text{the number of all } S \subseteq [d] \text{ satisfying } |S|=k \right) \cdot (d - 2k)^n \]

\[ = \sum_{k=0}^{d} \left( \text{the number of all } k\text{-element subsets of } [d] = \binom{d}{k} \right) \cdot (d - 2k)^n \]

\[ = \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n. \]

Hence, (23) becomes

\[ \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (e_1 + e_2 + \cdots + e_d)^n \]

\[ = \sum_{(e_1, e_2, \ldots, e_d) \in \{-1, 1\}^d} (d - 2 \cdot |\{i \in [d] \mid e_i = -1\}|)^n = \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n. \]

Comparing this with (22), we obtain

\[ \text{(the number of all all-even } (x_1, x_2, \ldots, x_n) \in [d]^n \text{)} \cdot 2^d = \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n. \]
Solving this for \((x_1, x_2, \ldots, x_n) \in [d]^n\), we obtain

\[
(\text{the number of all all-even } (x_1, x_2, \ldots, x_n) \in [d]^n) = \frac{1}{2d} \sum_{k=0}^{d} \binom{d}{k} (d - 2k)^n.
\]

This solves the exercise.

REFERENCES

