Math 5705: Enumerative Combinatorics,  
Fall 2018: Homework 5

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due date: **Wednesday, 28 November 2018** at the beginning of class,  
or before that by email or canvas.  
Please solve **at most 4 of the 6 exercises**!

1 **EXERCISE 1**

1.1 **PROBLEM**

A *point* shall mean an element of $\mathbb{Z}^2$, that is, a pair of integers. We depict these points as lattice points on the Cartesian plane, and add and subtract them as vectors.

Recall the notion of a *lattice path*, defined in §6.1 [class notes from 2018-11-12] and (equivalently) in UMN Spring 2018 Math 4707 Midterm 1 [Lattice paths have up-steps and right-steps.]

Fix a positive integer $k$.

We say that a point $(x, y) \in \mathbb{Z}^2$ is *off-limits* if $ky > x$. (Thus, the off-limits points are the ones that lie strictly above the $x = ky$ diagonal in Cartesian coordinates.)

A lattice path $(v_0, v_1, \ldots, v_n)$ is said to be *k-legal* if none of the points $v_0, v_1, \ldots, v_n$ is off-limits. Equivalently, a lattice path $(v_0, v_1, \ldots, v_n)$ is k-legal if each point $(x, y) \in \{v_0, v_1, \ldots, v_n\}$ satisfies $x \geq ky$. 

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For example, the lattice path from \((0, 0)\) to \((5, 2)\) drawn in the picture is not 2-legal, since it contains the off-limits point \((1, 1)\). Meanwhile, the lattice path from \((0, 0)\) to \((5, 2)\) drawn in the picture is 2-legal.

For any \(n \in \mathbb{Z}\) and \(m \in \mathbb{Z}\), we let \(L_{n,m,k}\) be the number of all \(k\)-legal lattice paths from \((0, 0)\) to \((n, m)\).

(a) Prove that \(L_{n,m,k} = L_{n-1,m,k} + L_{n,m-1,k}\) for any \(n \in \mathbb{Z}\) and \(m \in \mathbb{Z}\) satisfying \(n \geq km\) and \((n, m) \neq (0, 0)\).

(b) Prove that
\[
L_{n,m,k} = \binom{n+m}{m} - k \binom{n+m}{m-1}
\]
for all \(n \in \mathbb{N}\) and \(m \in \mathbb{N}\) satisfying \(n \geq km - 1\).

[The requirement \(n \geq km - 1\) as opposed to \(n \geq km\) is not a typo; the equality still holds for \(n = km - 1\), albeit for fairly simple reasons.]

(c) Prove that \(L_{n,m,k} = \frac{n+1-km}{n+1} \binom{n+m}{m}\) for all \(n \in \mathbb{N}\) and \(m \in \mathbb{N}\) satisfying \(n \geq km - 1\).

(d) Prove that
\[
L_{km,m,k} = \frac{1}{km+1} \binom{(k+1)m}{m} = \frac{1}{(k+1)m+1} \binom{(k+1)m+1}{m}
\]
for any \(m \in \mathbb{N}\).

1.2 Remark

This exercise generalizes Exercise 2 from UMN Spring 2018 Math 4707 Midterm 2 (except that I’ve added an extra equality to part (d)). You are free to solve it by copy-pasting the solution of the latter (download its TeX source—alas, computer-generated), or by referencing the solution of the latter and pointing out what changes are necessary and where.

1 Formally speaking, this lattice path is the list
\(((0, 0), (1, 0), (1, 1), (2, 1), (3, 1), (4, 1), (4, 2), (5, 2))\).
1.3 Solution

[...]

2 Exercise 2

2.1 Problem

We shall abbreviate “lattice path” as “LP”.

Recall that an LP is said to be legal if it is $k$-legal for $k = 1$.

Recall also that $C_n$ denotes the $n$-th Catalan number (that is, $\frac{1}{n+1} \binom{2n}{n}$) for any $n \in \mathbb{N}$.

If $v = (v_0, v_1, \ldots, v_n)$ is an LP, then an inversion of $v$ means a pair $(i, j) \in [n]^2$ such that $i < j$ and such that the $i$-th step of $v$ is an up-step (i.e., we have $v_i - v_{i-1} = (0, 1)$) but the $j$-th step of $v$ is a right-step (i.e., we have $v_j - v_{j-1} = (1, 0)$).

For example, the LP depicted in

![LP Diagram]

has the 5 inversions (2, 3) and (2, 4) and (2, 7) and (5, 7) and (6, 7) (and no others). It is easy to see that these inversions correspond to the 5 green squares under the LP in the above picture; more generally, any LP $v = (v_0, v_1, \ldots, v_n)$ from a point $s$ to a point $t$ subdivides its “bounding box” (i.e., the rectangle with opposing corners $s$ and $t$) into two parts, and the area of the part under the LP is exactly the number of inversions of $v$.

If $v = (v_0, v_1, \ldots, v_n)$ is a LP, then $\text{inv} \ v$ denotes the number of inversions of $v$.

Now, let $n \in \mathbb{N}$. Prove that

$$\sum_{v \text{ is a legal LP from } (0,0) \text{ to } (n,n)} (-1)^{\text{inv} \ v} = \begin{cases} C_{(n-1)/2}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

[Hint: What happens to a legal LP from $(0,0)$ to $(n,n)$ if we swap its 2-nd and 3-rd steps? If the answer is “nothing”, then what if we swap its 4-th and 5-th steps? If nothing again, its 6-th and 7-th steps?]

2.2 Solution

[...]

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3 EXERCISE 3

3.1 PROBLEM

Let \( x \in \mathbb{Q} \) and \( p \in \mathbb{N} \). Prove that

\[
\sum_{k=0}^{p} C_{k} \left( \frac{x - 2k}{p - k} \right) = \left( \frac{x + 1}{p} \right).
\]

[Hint: By the “polynomial identity trick”, it suffices to prove this in the case when \( x + 1 \geq 2p \).]

3.2 SOLUTION

[...]

4 EXERCISE 4

4.1 PROBLEM

Let \( n \) be a positive integer. Prove that

\[
\sum_{k=0}^{n} (-1)^{k} C_{k} \left( \frac{n + k}{2k} \right) = 0.
\]

4.2 SOLUTION

[...]

5 EXERCISE 5

5.1 PROBLEM

Let \( n \in \mathbb{N} \). A deranged involution of \([2n]\) shall mean a fixed-point-free involution \( \sigma : [2n] \to [2n] \) such that every \( i \in [n] \) satisfies \( \sigma (2i - 1) \neq 2i \).

(For example, the permutation of \([6]\) whose one-line notation is \([4, 5, 6, 1, 2, 3]\) is a deranged involution, but the permutation of \([6]\) whose one-line notation is \([6, 5, 4, 3, 2, 1]\) is not.)

Let \( a_{n} \) be the number of deranged involutions of \([2n]\).

(a) Prove that \( a_{0} = 1 \) and \( a_{1} = 0 \) and \( a_{n} = 2 (n - 1) (a_{n-1} + a_{n-2}) \) for all \( n \geq 2 \).

(b) Prove that

\[
a_{n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1 \cdot 3 \cdot 5 \cdots (2k - 1))
\]

for all \( n \in \mathbb{N} \).
6 Exercise 6

6.1 Problem

Let \( n \in \mathbb{N} \). Let \( \sigma \in S_n \). Set \( h(\sigma) = \sum_{i \in [n]} |\sigma(i) - i| \).

(a) Prove that
\[
h(\sigma) = 2 \sum_{i \in [n]; \sigma(i) > i} (\sigma(i) - i) = 2 \sum_{i \in [n]; \sigma(i) < i} (i - \sigma(i)).
\]

(b) Prove that \( h(s_i \circ \sigma) \leq h(\sigma) + 2 \) for each \( i \in [n-1] \).

(c) Prove that
\[
h(\sigma)/2 \leq \ell(\sigma) \leq h(\sigma).
\]

[Hint: As mentioned, you are free to use previous homework sets and midterms.]

6.2 Solution

[...]