1 Exercise 1

1.1 Problem

Let $A$ and $B$ be two finite sets, and let $f : A \to B$ be a map.

(a) Prove that the number of maps $g : B \to A$ satisfying $f \circ g \circ f = f$ is

$$|A|^{(B \setminus f(A))} \prod_{b \in f(A)} |f^{-1}(b)|.$$

(Here and in the following, $f(A)$ denotes the set $\{f(a) \mid a \in A\}$, whereas $f^{-1}(b)$ denotes the set $\{a \in A \mid f(a) = b\}$.)

(b) Prove that the number of maps $g : B \to A$ satisfying $f \circ g \circ f = f$ and $g \circ f \circ g = g$ is

$$|f(A)|^{(B \setminus f(A))} \prod_{b \in f(A)} |f^{-1}(b)|.$$
[Hint: For part (a), observe that
\[ |A|^{|B \setminus f(A)|} \prod_{b \in f(A)} |f^{-1}(b)| = \prod_{b \in B} \begin{cases} |f^{-1}(b)|, & \text{if } b \in f(A); \\ |A|, & \text{if } b \notin f(A). \end{cases} \]

What does this suggest about the construction of such maps \( g \)?

1.2 REMARK

The maps \( g \) in part (a) are called “generalized inverses” of \( f \). The maps \( g \) in part (b) are called “reflexive generalized inverses” of \( f \). Note that one consequence of part (b) is that there is always at least one reflexive generalized inverse of \( f \) (unless \( A \) is empty).

One can similarly define generalized inverses for linear maps between vector spaces; the resulting notion is much more well-known and has books devoted to it (see the Wikipedia for an overview).

1.3 SOLUTION

[...]

2 EXERCISE 2

2.1 PROBLEM

Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \). Prove that
\[ (n + m) \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n+j}{j} = n. \]

[Hint: The fraction on the left hand side has too many \( j \)'s. Try to simplify it to get the number of \( j \)'s down to just 1 (not counting the exponent in \((-1)^j\)).]

2.2 SOLUTION

[...]

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3 Exercise 3

3.1 Problem
Let $n$ be a positive integer. Let $a_1, a_2, \ldots, a_n$ be $n$ integers. Let $F : \mathbb{Z} \to \mathbb{R}$ be any function. Prove that

$$F(\max \{a_1, a_2, \ldots, a_n\}) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} F(\min \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}).$$

[Hint: This generalizes Exercise 5 on Spring 2018 Math 4707 homework set #2. Will some of the solutions given there still apply to this generalization?]

3.2 Solution

[...]

4 Exercise 4

4.1 Problem
Recall once again the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$, which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2. \tag{1}$$

Now, let us define $f_n$ for negative integers $n$ as well, by “applying (1) backwards”: This means that we set $f_{n-2} = f_n - f_{n-1}$ for all integers $n \leq 1$. This allows us to recursively compute $f_{-1}, f_{-2}, f_{-3}, \ldots$ (in this order). For example,

$$f_{-1} = f_1 - f_0 = 1 - 0 = 1;$$
$$f_{-2} = f_0 - f_{-1} = 0 - 1 = -1;$$
$$f_{-3} = f_{-1} - f_{-2} = 1 - (-1) = 2,$$

etc.

(a) Prove that $f_{-n} = (-1)^{n-1} f_n$ for each $n \in \mathbb{Z}$.

(b) Prove that $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

(c) Prove that $7f_n = f_{n-4} + f_{n+4}$ for all $n \in \mathbb{Z}$.

[Hint: This is not an exercise about the combinatorial interpretations (domino tilings, lacunar subsets, etc.) of Fibonacci numbers. Make sure that your proofs cover all integers, not just elements of $\mathbb{N}$.]

4.2 Solution

[...]
5 Exercise 5

5.1 Problem
Let \( n \in \mathbb{N} \) and \( p \in \{0, 1, \ldots, n\} \). A \( p \)-derangement of \([n]\) shall mean a permutation \( \sigma \) of \([n]\) such that every \( i \in [n - p] \) satisfies \( \sigma(i) \neq i + p \). Compute the number of all \( p \)-derangements of \([n]\) as a sum of the form \( \sum_{i=0}^{n-p} \cdot \cdot \cdot \).

[Hint: The case \( p = 1 \) was Exercise 6 on Spring 2018 Math 4707 homework set #2.]

5.2 Solution
[...]

6 Exercise 6

6.1 Problem
Let \( n \) and \( k \) be positive integers. A \( k \)-smord will mean a \( k \)-tuple \( (a_1, a_2, \ldots, a_k) \in [n]^k \) such that no two consecutive entries of this \( k \)-tuple are equal (i.e., we have \( a_i \neq a_{i+1} \) for all \( i \in [k - 1] \)). For example, \((4, 1, 4, 2, 6)\) is a 5-smord (when \( n \geq 6 \)), but \((1, 4, 4, 2, 6)\) is not.

It is easy to see that the number of \( k \)-smords is \( n(n-1)^{k-1} \). (See, e.g., Exercise 5 on Math 4990 Fall 2017 homework set #3.)

A double \( k \)-smord shall mean a pair \(((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k))\) of two \( k \)-smords \((a_1, a_2, \ldots, a_k)\) and \((b_1, b_2, \ldots, b_k)\) such that every \( i \in [k] \) satisfies \( a_i \neq b_i \).

Prove that the number of double \( k \)-smords is \( n(n-1)(n^2-3n+3)^{k-1} \).

6.2 Remark
“Smord” is short for “Smirnov word” (which is how these tuples are sometimes called).

Double \( k \)-smords can also be regarded as \( 2 \times k \)-matrices with entries lying in \([n]\) and with the property that no two adjacent entries are equal. (The double \( k \)-smord \(((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k))\) thus corresponds to the \( 2 \times k \)-matrix \( \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix} \).)

6.3 Solution
[...]

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