Math 5705: Enumerative Combinatorics,
Fall 2018: Midterm 2

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due date: Wednesday, 14 November 2018 at the beginning of class,
or before that by email or canvas.
Please solve at most 4 of the 6 exercises!
Beware: Collaboration is not allowed on midterms!

NOTATIONS

Here is a list of notations that are used in this homework:

• We shall use the Iverson bracket notation as well as the notation \([n]\) for the set
\(\{1, 2, \ldots, n\}\) (when \(n \in \mathbb{Z}\)).

• If \(n \in \mathbb{N}\), then \(S_n\) denotes the set of all permutations of \([n]\).

• If \(n \in \mathbb{N}\) and \(\sigma \in S_n\), then:
  – a descent of the permutation \(\sigma\) denotes an element \(k \in [n - 1]\) satisfying \(\sigma(k) > \sigma(k + 1)\).
  – the descent set \(\text{Des} \sigma\) of \(\sigma\) is defined as the set of all descents of \(\sigma\).
  – the descent number \(\text{des} \sigma\) of \(\sigma\) is defined as the number of all descents of \(\sigma\) (that is, \(\text{des} \sigma = |\text{Des} \sigma|\)).
  – the one-line notation \(\text{OLN} \sigma\) of \(\sigma\) is defined as the \(n\)-tuple \((\sigma(1), \sigma(2), \ldots, \sigma(n))\).
    Often, this \(n\)-tuple is written with square brackets, i.e., as \([\sigma(1), \sigma(2), \ldots, \sigma(n)]\).
– for each $i \in [n]$, we define $\ell_i(\sigma)$ to be the number of all $j \in \{i + 1, i + 2, \ldots, n\}$ satisfying $\sigma(i) > \sigma(j)$.
– we say that $\sigma$ is 312-avoiding if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(j) < \sigma(k) < \sigma(i)$.
– we say that $\sigma$ is 321-avoiding if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(k) < \sigma(j) < \sigma(i)$.

• For any $n \in \mathbb{N}$ and any $i \in [n - 1]$, we let $s_i$ denote the permutation in $S_n$ that swaps $i$ with $i + 1$ while leaving all other elements of $[n]$ unchanged. (This assumes that $n$ is determined by the context.)

• For any $n \in \mathbb{N}$ and any $k$ distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $cyc_{i_1, i_2, \ldots, i_k}$ be the permutation in $S_n$ that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$ (respectively) while leaving all the other elements of $[n]$ unchanged. (Again, this relies on $n$ being clear from the context.)

• For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, the notation $\langle n \rangle_k$ denotes the number of all permutations $\sigma \in S_n$ having exactly $k$ descents. This is called an Eulerian number.

• If $X$ is a set, and if $\alpha : X \to X$ and $\beta : X \to X$ are two maps, then the composition $\alpha \circ \beta : X \to X$ is simply denoted by $\alpha \beta$, and is called the product of $\alpha$ and $\beta$. This notation is used for permutations, in particular.

• If $X$ is a set, if $k \in \mathbb{N}$, and if $f : X \to X$ is any map, then the map $f^k : X \to X$ is defined by

$$f^k = f \circ f \circ \cdots \circ f \quad (k \text{ times})$$

This map $f^k$ is called the $k$-th power of $f$ (or $k$-th composition power of $f$). These powers behave as one would expect as long as you have only one map $f : X \to X$ (meaning that $f^{a+b} = f^a f^b$ and $f^{ab} = (f^a)^b$ for any $a, b \in \mathbb{N}$); but be careful with several maps (e.g., two maps $f : X \to X$ and $g : X \to X$ don’t always satisfy $(fg)^a = f^a g^a$). See [Grinb16, Section 2.13.8] for details (where I write $f^{\circ k}$ instead of $f^k$).

• If $X$ is a set, and if $f : X \to X$ is a map, then:
  – we say that $f$ is an involution if and only if $f^2 = \text{id}$. (Note that every involution is automatically a permutation.)
  – we say that $f$ is fixed-point-free if each $x \in X$ satisfies $f(x) \neq x$ (that is, if $f$ has no fixed points). (Note that the fixed-point-free permutations are precisely the derangements.)
1 EXERCISE 1

1.1 PROBLEM

Let $n$ and $k$ be positive integers.

For each $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$, we let $\tau^i_\leftarrow \in S_n$ be the permutation such that

$$\text{OLN} (\tau^i_\leftarrow) = (\tau(1), \tau(2), \ldots, \tau(i), n, \tau(i+1), \tau(i+2), \ldots, \tau(n-1))$$

(that is, $\text{OLN} (\tau^i_\leftarrow)$ is obtained from $\text{OLN} \tau$ by inserting an $n$ right after the $i$-th entry).

(a) Prove that each $i \in \{0, 1, \ldots, n-1\}$ and $\tau \in S_{n-1}$ satisfy

$$[\text{des} (\tau^i_\leftarrow) = k] = [\text{des} \tau = k - 1 \text{ and } \tau(i) < \tau(i+1)] + [\text{des} \tau = k \text{ and } \tau(i) > \tau(i+1)],$$

where we set $\tau(0) = 0$ and $\tau(n) = 0$.

(b) Prove that the map

$$\{0, 1, \ldots, n-1\} \times S_{n-1} \to S_n,$$

$$(i, \tau) \mapsto \tau^i_\leftarrow$$

is a bijection.

(c) Prove that

$$\begin{bmatrix} n \\ k \end{bmatrix} = (k+1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + (n-k) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. $$

[Hint: You don’t need to write more than a few sentences for parts (a) and (b).]

1.2 SOLUTION

[...]

2 EXERCISE 2

2.1 PROBLEM

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. For each $i \in [n]$, let

$$a_i = \text{cyc}_{i', i'-1, \ldots, i} = s_{i'-1} s_{i'-2} \cdots s_i \in S_n, \quad \text{where } i' = i + \ell_i(\sigma).$$

Prove that $\sigma = a_1 a_2 \cdots a_n$.

[Hint: Prove, “more generally”, that if $j \in \{0, 1, \ldots, n\}$ is such that $1, 2, \ldots, j$ are fixed points of $\sigma$, then $\sigma = a_{j+1} a_{j+2} \cdots a_n$.]
2.2 Solution

[...]

3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$.

(a) Prove that any $\sigma \in S_n$ and any $i \in [n]$ satisfy $\sigma(i) \leq i + \ell_i(\sigma)$.

(b) Prove that, for a given $\sigma \in S_n$, the following three statements are equivalent:
   
   $A$: We have $\sigma(i) \leq i + 1$ for all $i \in [n-1]$.
   
   $B$: The permutation $\sigma$ is both 321-avoiding and 312-avoiding.
   
   $C$: We have $\ell_i(\sigma) \in \{0, 1\}$ for each $i \in [n]$. (In other words, the Lehmer code of $\sigma$
   consists only of 0’s and 1’s.)

(c) Assuming that $n \geq 1$, prove that the number of $\sigma \in S_n$ satisfying these three state-ements is $2^{n-1}$.

3.2 Solution

[...]

4 Exercise 4

4.1 Problem

Let $n \geq 2$, and set $S = [n]$. Let $i \in [n-1]$. Prove that:

(a) The number of maps $f : S \to S$ with $f(i) = n$ and $f^n(S) = \{n\}$ is $2n^{n-3}$.

(b) Let $j \in [n-1]$ be such that $i \neq j$. The number of maps $f : S \to S$ with $f(i) = j$ and $f^n(S) = \{n\}$ is $n^{n-3}$.

[Hint: Substitute appropriate numbers for the variables in the Matrix-Tree Theorem.]

4.2 Solution

[...]
5 Exercise 5

5.1 Problem

(a) For each $n \in \mathbb{N}$, prove that the number of fixed-point-free involutions $[n] \rightarrow [n]$ is

\[
\begin{cases} 
  1 \cdot 3 \cdot 5 \cdots (n-1), & \text{if } n \text{ is even;} \\
  0, & \text{if } n \text{ is odd.}
\end{cases}
\]

(b) For each $n \in \mathbb{N}$, we let $t_n$ be the number of all involutions in $S_n$. Prove that

\[
t_n = \sum_{k=0}^{n} \binom{n}{2k} (1 \cdot 3 \cdot 5 \cdots (2k-1))
\]

for each $n \in \mathbb{N}$.

(c) For each $n \in \mathbb{N}$, prove that the number of maps $f : [n] \rightarrow [n]$ satisfying $f^3 = f$ is

\[
\sum_{k=0}^{n} \binom{n}{k} k^{n-k} k!.
\]

5.2 Remark

The numbers in part (a) form the sequence A123023 in the OEIS (And if you omit the terms for odd $n$, which are all zero, then you obtain sequence A001147, known as the double factorials.)

The numbers $t_0, t_1, t_2, \ldots$ in part (b) are sometimes called the telephone numbers, because an involution in $S_n$ is a way how phone calls can be happening between $n$ people $1, 2, \ldots, n$, assuming there are no conference calls. This is sequence A000085 in the OEIS.

Finally, the numbers in part (c) form sequence A060905.

5.3 Solution

[...]

6 Exercise 6

6.1 Problem

Let $n$ be a positive integer, and let $p \in \{0, 1, \ldots, n\}$.

A permutation $\sigma \in S_n$ shall be called a $p$-desarrangement if it satisfies

either $\sigma = \text{id}$ and $2 \mid n$ or $\sigma(1) \leq p$ or $\sigma \neq \text{id}$ and $2 \mid \min (\Des \sigma)$.

(The condition $2 \mid \min (\Des \sigma)$ means that the smallest descent of $\sigma$ is even.\footnote{This is well-defined, since $\sigma \neq \text{id}$ shows that $\sigma$ has at least one descent. Further $p$-desarrangements are id when $n$ is even, and all permutations starting with a number $\leq p$ (in one-line notation).} This is well-defined, since $\sigma \neq \text{id}$ shows that $\sigma$ has at least one descent. Further $p$-desarrangements are id when $n$ is even, and all permutations starting with a number $\leq p$ (in one-line notation).)

Prove that the number of $p$-desarrangements in $S_n$ is

\[
\sum_{k=0}^{n-p} \binom{n-p}{k} \cdot (-1)^k (n-k)!.
\]
6.2 Remark

This number is exactly the number of $p$-derangements in $S_n$, as defined in Exercise 5 of midterm #1. This suggests the existence of a bijection between the $p$-derangements and the $p$-derangements. Such a thing has indeed been found in the case when $p = 0$. In this case, the 0-derangements are known as desarrangements (a pun on the name and the word “derangement”), whereas the 0-derangements are precisely the derangements. The desarrangements are just the permutations $\sigma \in S_n$ satisfying either $\sigma = id$ or $(\sigma \neq id$ and $2 \mid \min (\text{Des} \sigma))$. One known bijection between the derangements and the desarrangements proceeds as follows:

- Let $\sigma \in S_n$ be a derangement. We want to define the corresponding desarrangement $F(\sigma)$.
- Compute the disjoint cycle decomposition of $\sigma$, and write it in such a way that each cycle contains its largest entry in its second position, and that the cycles are ordered in increasing order of their largest entries. That is, write

  $$\sigma = \text{cyc}_{a_1,1,a_1,2,\ldots,a_1,n_1}\text{cyc}_{a_2,1,a_2,2,\ldots,a_2,n_2}\ldots\text{cyc}_{a_k,1,a_k,2,\ldots,a_k,n_k},$$

  where each of the numbers $1, 2, \ldots, n$ appears exactly once among the $a_{i,j}$, and where

  $$a_{i,2} \geq a_{i,j} \quad \text{for all } i \text{ and } j, \quad \text{and} \quad a_{1,2} < a_{2,2} < \cdots < a_{k,2}.$$

- Now, let $F(\sigma)$ be the permutation whose one-line notation is

  $$(a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}, \ldots, a_{k,1}, a_{k,2}, \ldots, a_{k,n_k}).$$

For example, if $n = 7$ and $\sigma = [5, 3, 7, 6, 1, 4, 2]$ in one-line notation, then the appropriate representation of $\sigma$ is $\sigma = \text{cyc}_{1,5}\text{cyc}_{4,6}\text{cyc}_{3,7,2}$ and thus $F(\sigma) = [1, 5, 4, 6, 3, 7, 2]$ in one-line notation.

It is far from trivial to check that this is actually a well-defined bijection. I don’t know if anything like that exists for $p \neq 0$. Feel free to explore. (But the simplest way to solve the exercise is not by bijection.)

6.3 Solution

[...]

underline marking the position of the smallest descent):

$$[1, 2, 3, 5, 4], \quad [1, 2, 4, 5, 3], \quad [1, 3, 2, 4, 5], \quad [1, 3, 2, 5, 4], \quad [1, 3, 4, 5, 2],$$

$$[1, 4, 2, 3, 5], \quad [1, 2, 5, 3], \quad [1, 4, 3, 2, 5], \quad [1, 4, 3, 5, 2], \quad [1, 5, 2, 3, 4],$$

$$[1, 5, 2, 4, 3], \quad [1, 5, 3, 2, 4], \quad [1, 5, 3, 4, 2], \quad [1, 5, 4, 2, 3], \quad [1, 5, 4, 3, 2],$$

$$[2, 3, 1, 4, 5], \quad [2, 3, 1, 5, 4], \quad [2, 3, 4, 5, 1], \quad [2, 4, 1, 3, 5], \quad [2, 4, 1, 5, 3],$$

$$[2, 4, 3, 1, 5], \quad [2, 4, 3, 5, 1], \quad [2, 5, 1, 3, 4], \quad [2, 5, 1, 4, 3], \quad [2, 5, 3, 1, 4],$$

$$[2, 5, 3, 4, 1], \quad [2, 5, 4, 1, 3], \quad [2, 5, 4, 3, 1], \quad [3, 4, 1, 2, 5], \quad [3, 4, 1, 5, 2],$$

$$[3, 4, 2, 1, 5], \quad [3, 4, 2, 5, 1], \quad [3, 5, 1, 2, 4], \quad [3, 5, 1, 4, 2], \quad [3, 5, 2, 1, 4],$$

$$[3, 5, 2, 4, 1], \quad [3, 5, 4, 1, 2], \quad [3, 5, 4, 2, 1], \quad [4, 5, 1, 2, 3], \quad [4, 5, 1, 3, 2],$$

$$[4, 5, 2, 1, 3], \quad [4, 5, 2, 3, 1], \quad [4, 5, 3, 1, 2], \quad [4, 5, 3, 2, 1].$$
REFERENCES


The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10.