due date: **Wednesday, 14 November 2018** at the beginning of class, 
or before that by email or canvas. 
Please solve **at most 4 of the 6 exercises**! 
Beware: **Collaboration is not allowed** on midterms!

**Notations**

Here is a list of notations that are used in this homework:

- We shall use the Iverson bracket notation as well as the notation $[n]$ for the set 
  $\{1, 2, \ldots, n\}$ (when $n \in \mathbb{Z}$).

- If $n \in \mathbb{N}$ and $\sigma \in S_n$, then:
  - a **descent** of the permutation $\sigma$ denotes an element $k \in [n-1]$ satisfying $\sigma(k) > \sigma(k+1)$.
  - the **descent set** $\text{Des} \, \sigma$ of $\sigma$ is defined as the set of all descents of $\sigma$.
  - the **descent number** $\text{des} \, \sigma$ of $\sigma$ is defined as the number of all descents of $\sigma$ (that is, $\text{des} \, \sigma = |\text{Des} \, \sigma|$).
  - the **one-line notation** $\text{OLN} \, \sigma$ of $\sigma$ is defined as the $n$-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. 
    Often, this $n$-tuple is written with square brackets, i.e., as $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$.
  - for each $i \in [n]$, we define $\ell_i(\sigma)$ to be the number of all $j \in \{i+1, i+2, \ldots, n\}$ 
    satisfying $\sigma(i) > \sigma(j)$. 


we say that $\sigma$ is $312$-avoiding if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(j) < \sigma(k) < \sigma(i)$.

we say that $\sigma$ is $321$-avoiding if there exist no three elements $i, j, k \in [n]$ satisfying $i < j < k$ and $\sigma(k) < \sigma(j) < \sigma(i)$.

For any $n \in \mathbb{N}$ and any $i \in [n - 1]$, we let $s_i$ denote the permutation in $S_n$ that swaps $i$ with $i + 1$ while leaving all other elements of $[n]$ unchanged. (This assumes that $n$ is determined by the context.)

For any $n \in \mathbb{N}$ and any $k$ distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\text{cyc}_{i_1, i_2, \ldots, i_k}$ be the permutation in $S_n$ that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$ (respectively) while leaving all the other elements of $[n]$ unchanged. (Again, this relies on $n$ being clear from the context.)

For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, the notation $\langle n \rangle_k$ denotes the number of all permutations $\sigma \in S_n$ having exactly $k$ descents. This is called an Eulerian number.

If $X$ is a set, and if $\alpha : X \to X$ and $\beta : X \to X$ are two maps, then the composition $\alpha \circ \beta : X \to X$ is simply denoted by $\alpha \beta$, and is called the product of $\alpha$ and $\beta$. This notation is used for permutations, in particular.

If $X$ is a set, if $k \in \mathbb{N}$, and if $f : X \to X$ is any map, then the map $f^k : X \to X$ is defined by

$$f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}} = \underbrace{f f \cdots f}_{k \text{ times}}.$$  

This map $f^k$ is called the $k$-th power of $f$ (or $k$-th composition power of $f$). These powers behave as one would expect as long as you have only one map $f : X \to X$ (meaning that $f^{a+b} = f^a f^b$ and $f^{ab} = (f^a)^b$ for any $a, b \in \mathbb{N}$); but be careful with several maps (e.g., two maps $f : X \to X$ and $g : X \to X$ don’t always satisfy $(fg)^a = f^a g^a$). See [Grinbe16, Section 2.13.8] for details (where I write $f^k$ instead of $f^k$).

If $X$ is a set, and if $f : X \to X$ is a map, then:

- we say that $f$ is an involution if and only if $f^2 = \text{id}$. (Note that every involution is automatically a permutation.)
- we say that $f$ is fixed-point-free if each $x \in X$ satisfies $f(x) \neq x$ (that is, if $f$ has no fixed points). (Note that the fixed-point-free permutations are precisely the derangements.)

1 Exercise 1

1.1 Problem

Let $n$ and $k$ be positive integers.
For each $i \in \{0, 1, \ldots, n - 1\}$ and $\tau \in S_{n-1}$, we let $\tau^i_\leftarrow \in S_n$ be the permutation such that
\[
\text{OLN} (\tau^i_\leftarrow) = (\tau(1), \tau(2), \ldots, \tau(i), n, \tau(i + 1), \tau(i + 2), \ldots, \tau(n - 1))
\]
(that is, $\text{OLN} (\tau^i_\leftarrow)$ is obtained from $\text{OLN} \tau$ by inserting an $n$ right after the $i$-th entry).

(a) Prove that each $i \in \{0, 1, \ldots, n - 1\}$ and $\tau \in S_{n-1}$ satisfy
\[
[\text{des} (\tau^i_\leftarrow) = k] = [\text{des} \tau = k - 1 \text{ and } \tau(i) < \tau(i + 1)] + [\text{des} \tau = k \text{ and } \tau(i) > \tau(i + 1)],
\]
where we set $\tau(0) = 0$ and $\tau(n) = 0$.

(b) Prove that the map
\[
\{0, 1, \ldots, n - 1\} \times S_{n-1} \to S_n,
\]
\[
(i, \tau) \mapsto \tau^i_\leftarrow
\]
is a bijection.

(c) Prove that
\[
\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle = (k + 1) \left\langle \begin{array}{c} n - 1 \\ k \end{array} \right\rangle + (n - k) \left\langle \begin{array}{c} n - 1 \\ k - 1 \end{array} \right\rangle.
\]

[Hint: You don’t need to write more than a few sentences for parts (a) and (b).]

1.2 SOLUTION

[...]

2 EXERCISE 2

2.1 PROBLEM

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. For each $i \in [n]$, let
\[
a_i = \text{cyc}_{i', i'-1, \ldots, i} = s_{i'-1}s_{i'-2} \cdots s_i \in S_n, \quad \text{where } i' = i + \ell_i(\sigma).
\]

Prove that $\sigma = a_1a_2 \cdots a_n$.

[Hint: Prove, “more generally”, that if $j \in \{0, 1, \ldots, n\}$ is such that $1, 2, \ldots, j$ are fixed points of $\sigma$, then $\sigma = a_{j+1}a_{j+2} \cdots a_n$.]

2.2 SOLUTION

[...]

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3 EXERCISE 3

3.1 PROBLEM

Let \( n \in \mathbb{N} \).

(a) Prove that any \( \sigma \in S_n \) and any \( i \in [n] \) satisfy \( \sigma(i) \leq i + \ell_i(\sigma) \).

(b) Prove that, for a given \( \sigma \in S_n \), the following three statements are equivalent:

\( A \): We have \( \sigma(i) \leq i + 1 \) for all \( i \in [n-1] \).

\( B \): The permutation \( \sigma \) is both 321-avoiding and 312-avoiding.

\( C \): We have \( \ell_i(\sigma) \in \{0, 1\} \) for each \( i \in [n] \). (In other words, the Lehmer code of \( \sigma \) consists only of 0’s and 1’s.)

(c) Assuming that \( n \geq 1 \), prove that the number of \( \sigma \in S_n \) satisfying these three statements is \( 2^{n-1} \).

3.2 SOLUTION

[...]

4 EXERCISE 4

4.1 PROBLEM

Let \( n \geq 2 \), and set \( S = [n] \). Let \( i \in [n-1] \). Prove that:

(a) The number of maps \( f : S \to S \) with \( f(i) = n \) and \( f^n(S) = \{n\} \) is \( 2n^{n-3} \).

(b) Let \( j \in [n-1] \) be such that \( i \neq j \). The number of maps \( f : S \to S \) with \( f(i) = j \) and \( f^n(S) = \{n\} \) is \( n^{n-3} \).

[Hint: Substitute appropriate numbers for the variables in the Matrix-Tree Theorem.]

4.2 SOLUTION

[...]

5 EXERCISE 5

5.1 PROBLEM

(a) For each \( n \in \mathbb{N} \), prove that the number of fixed-point-free involutions \( [n] \to [n] \) is

\[
\begin{cases}
1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-1), & \text{if } n \text{ is even;} \\
0, & \text{if } n \text{ is odd.}
\end{cases}
\]
(b) For each \( n \in \mathbb{N} \), we let \( t_n \) be the number of all involutions in \( S_n \). Prove that
\[
t_n = \sum_{k=0}^{n} \binom{n}{2k} (1 \cdot 3 \cdot 5 \cdots (2k - 1)) \quad \text{for each } n \in \mathbb{N}.
\]

(c) For each \( n \in \mathbb{N} \), prove that the number of maps \( f : [n] \to [n] \) satisfying \( f^3 = f \) is
\[
\sum_{k=0}^{n} \binom{n}{k} k^{n-k} t_k.
\]

5.2 REMARK

The numbers in part (a) form the sequence A123023 in the OEIS. (And if you omit the terms for odd \( n \), which are all zero, then you obtain sequence A001147, known as the double factorials.)

The numbers \( t_0, t_1, t_2, \ldots \) in part (b) are sometimes called the telephone numbers, because an involution in \( S_n \) is a way how phone calls can be happening between \( n \) people \( 1, 2, \ldots, n \), assuming there are no conference calls. This is sequence A000085 in the OEIS.

Finally, the numbers in part (c) form sequence A060905.

5.3 SOLUTION

[...]

6 EXERCISE 6

6.1 PROBLEM

Let \( n \) be a positive integer, and let \( p \in \{0, 1, \ldots, n\} \).

A permutation \( \sigma \in S_n \) shall be called a \( p \)-desarrangement if it satisfies

either \( \sigma = \text{id} \) or \( \sigma(1) \leq p \) or \( \sigma \neq \text{id} \) and \( 2 \mid \min(\text{Des} \sigma) \).

(The condition \( 2 \mid \min(\text{Des} \sigma) \) means that the smallest descent of \( \sigma \) is even.\(^1\) This is well-defined, since \( \sigma \neq \text{id} \) shows that \( \sigma \) has at least one descent. Further \( p \)-desarrangements are id and all permutations starting with a number \( \leq p \) (in one-line notation).)

\(^1\)Here are all permutations \( \sigma \neq \text{id} \) in \( S_5 \) that satisfy this condition (written in one-line notation, with an underline marking the position of the smallest descent):

\[
\begin{align*}
[1, 2, 3, 5, 4], & \quad [1, 2, 4, 5, 3] , \quad [1, 2, 5, 4], \quad [1, 2, 5, 4, 3], \quad [1, 3, 4, 5, 2], \\
[1, 4, 2, 3, 5], & \quad [1, 4, 2, 5, 3], \quad [1, 4, 3, 2, 5], \quad [1, 4, 3, 5, 2], \quad [1, 5, 2, 3, 4], \\
[1, 5, 2, 4, 3], & \quad [1, 5, 3, 2, 4], \quad [1, 5, 3, 4, 2], \quad [1, 5, 4, 2, 3], \quad [1, 5, 4, 3, 2], \\
[2, 3, 4, 5], & \quad [2, 3, 4, 5, 1], \quad [2, 3, 4, 5, 1], \quad [2, 4, 1, 3, 5], \quad [2, 4, 1, 5, 3], \\
[2, 4, 3, 1, 5], & \quad [2, 4, 3, 5, 1], \quad [2, 5, 1, 3, 4], \quad [2, 5, 1, 4, 3], \quad [2, 5, 3, 1, 4], \\
[2, 5, 3, 4, 1], & \quad [2, 5, 4, 1, 3], \quad [2, 5, 4, 3, 1], \quad [3, 4, 1, 2, 5], \quad [3, 4, 1, 5, 2], \\
[3, 4, 2, 1, 5], & \quad [3, 4, 2, 5, 1], \quad [3, 5, 1, 2, 4], \quad [3, 5, 1, 4, 2], \quad [3, 5, 2, 1, 4], \\
[3, 5, 2, 4, 1], & \quad [3, 5, 2, 4, 1], \quad [3, 5, 4, 2, 1], \quad [4, 5, 1, 2, 3], \quad [4, 5, 1, 3, 2], \\
[4, 5, 2, 1, 3], & \quad [4, 5, 2, 3, 1], \quad [4, 5, 3, 1, 2], \quad [4, 5, 3, 2, 1].
\end{align*}
\]
Prove that the number of \( p \)-desarrangements in \( S_n \) is
\[
\sum_{k=0}^{n-p} \binom{n-p}{k} \cdot (-1)^k (n-k)!.
\]

6.2 Remark

This number is exactly the number of \( p \)-derangements in \( S_n \), as defined in Exercise 5 of midterm \#1. This suggests the existence of a bijection between the \( p \)-desarrangements and the \( p \)-derangements. Such a thing has indeed been found in the case when \( p = 0 \). In this case, the 0-desarrangements are known as \textit{desarrangements} (a pun on the name Désarmenien and the word “derangement”), whereas the 0-derangements are precisely the derangements. The desarrangements are just the permutations \( \sigma \in S_n \) satisfying either \( \sigma = \text{id} \) or \( (\sigma \neq \text{id} \text{ and } 2 \mid \min (\text{Des} \sigma)) \). One known bijection between the derangements and the desarrangements proceeds as follows:

- Let \( \sigma \in S_n \) be a derangement. We want to define the corresponding desarrangement \( F(\sigma) \).
- Compute the disjoint cycle decomposition of \( \sigma \), and write it in such a way that each cycle contains its \textbf{largest} entry in its \textbf{second} position, and that the cycles are ordered in \textbf{increasing order of their largest entries}. That is, write
  \[
  \sigma = \text{cyc}_{a_{1,1},a_{1,2},\ldots,a_{1,n_1}} \text{cyc}_{a_{2,1},a_{2,2},\ldots,a_{2,n_2}} \cdots \text{cyc}_{a_{k,1},a_{k,2},\ldots,a_{k,n_k}},
  \]
  where each of the numbers \( 1, 2, \ldots, n \) appears exactly once among the \( a_{i,j} \), and where \( a_{i,2} \geq a_{i,j} \) for all \( i \) and \( j \), and \( a_{1,2} < a_{2,2} < \cdots < a_{k,2} \).
- Now, let \( F(\sigma) \) be the permutation whose one-line notation is
  \[
  (a_{1,1},a_{1,2},\ldots,a_{1,n_1},a_{2,1},a_{2,2},\ldots,a_{2,n_2},\ldots,a_{k,1},a_{k,2},\ldots,a_{k,n_k}).
  \]

For example, if \( n = 7 \) and \( \sigma = [5, 3, 7, 6, 1, 4, 2] \) in one-line notation, then the appropriate representation of \( \sigma \) is \( \sigma = \text{cyc}_{1,5} \text{cyc}_{4,6} \text{cyc}_{3,7,2} \) and thus \( F(\sigma) = [1, 5, 4, 6, 3, 7, 2] \) in one-line notation.

It is far from trivial to check that this is actually a well-defined bijection. I don’t know if anything like that exists for \( p \neq 0 \). Feel free to explore. (But the simplest way to solve the exercise is not by bijection.)

6.3 Solution

[...]

References


\url{http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf}

The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see \url{https://github.com/darijgr/detnotes/releases/tag/2018-10-03}.

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