One more fact about tournaments:

Prop. 1. Let $D = (V, A)$ be a tournament.

Let $(u, v, w)$ be a 3-cycle of $D$.

(A "3-cycle" of $D$ means a triple $(u, v, w)$ of vertices of $D$ such that $uv, vw, wu \in A$.)

Let $D'$ be the tournament obtained from $D$ by reorienting the arcs $uv, vw, wu$ (this means replacing them by $vu, wv, uv$).

Then, \# of 3-cycles of $D'$
\quad = \# of 3-cycles of $D$.

Let's give two proofs:

1st proof. HW2 exercise 5(b)

shows that the \# of 3-cycles of $D$ depends only on $|V|$ and the degrees $\deg(v)$ of the vertices $v \in V$. But these do not change when we reorient our arcs $uv, vw, wu$ (since each of $u, v, w$
loses 1 outgoing arc and gains another). Hence, \# of 3-cycles also doesn’t change. \(\square\)

2nd proof. It is easy to prove the claim in the case \(|V| \leq 4\) (just check all cases). Hence, \(\forall x \in V\), we have:

(1) \# of \# \# of 3-cycles of \(D'\) whose vertices belong to \(\{u, v, w, x\}\)

\[= \# \text{ of } 3\text{-cycles of } D \text{ whose vertices belong to } \{u, v, w, x\}\]

(because the induced subdigraph on the subset \(\{u, v, w, x\}\) of a tournament is again a tournament).

Now, the 3-cycles of \(D\) can be of the following 3 types:

Type 1: 3-cycles that contain at most 1 of the vertices \(u, v, w, x\).

Type 2: 3-cycles that contain at most 2 precisely 2 of the vertices \(u, v, w, x\).
Type 3: 3-cycles that contain all of the vertices \( u, v, w \).

The 3-cycles of Type 2 can be classified further: Each of them has

Type 2\(_x\): 3-cycles that contain precisely 2 of the vertices \( u, v, w \), and also the vertex \( x \) for a unique \( x \in V \setminus \{u, v, w\} \).

Now,

\[
\text{# of 3-cycles of } D' \text{ of Type 1} = \text{# of 3-cycles of } D \text{ of Type 1}
\]

(since 3-cycles of Type 1 are preserved when we reorient arcs \( uv, vu, wu \));

\[
\text{# of 3-cycles of } D' \text{ of Type } 2_x = \text{# of 3-cycles of } D \text{ of Type } 2_x \quad \forall x \in V \setminus \{u, v, w\}
\]

(by (1)).
\# of 3-cycles of \textit{Type D'} of Type 3
\hspace{1cm} = \# of 3-cycles of \textit{D} of Type 3
(since both numbers are divisible by 3),

Adding these equalities together, we get
\# of 3-cycles of \textit{D'}
\hspace{1cm} = \# of 3-cycles of \textit{D}.

\hspace{1cm}

Reorienting the arcs \textit{uc}, \textit{uc}, \textit{uc}.
Now, a few reminders about permutations.

**Def.** A permutation of a set $X$ is a bijection $X \to X$.

**Def.** For each $n \in \mathbb{N}$, we let $S_n$ be the set of all permutations of \{1, 2, ..., $n$\}. Note that $|S_n| = n!$.

There are several ways to write a permutation $\sigma \in S_n$:

- as the $n$-tuple $[\sigma(1), \sigma(2), ..., \sigma(n)]$ ("one-line notation").
- as the $2 \times n$-table $(\begin{array}{cccc} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{array})$ ("two-line notation").

- as a digraph

$$\left( \{1, 2, ..., n\} \right. \left. , \{(i, \sigma(i)) \mid i \in \{1, 2, ..., n\}\} \right)$$

**Examples:** Let $\sigma$ be the permutation of \{1, 2, 3, 4, 5\} starting sending

$1 \to 3$, $2 \to 2$, $3 \to 6$, $4 \to 5$, $5 \to 4$, $6 \to 1$. 

Then,

- the one-line notation for \( \sigma \) is \([3, 2, 6, 5, 4, 1]\),
- the two-line notation for \( \sigma \) is
  \[
  \begin{pmatrix}
    1 & 2 & 3 & 4 & 5 & 6 \\
    3 & 2 & 6 & 5 & 4 & 1
  \end{pmatrix}
  \]
- the digraph for \( \sigma \) is

![Digraph](image)

Rmk: The digraph for \( \sigma \in S_6 \) has the property that for any vertex \( \nu \) we have

\[
\text{deg}^+ \nu = 3 \quad \text{and} \quad \text{deg}^- \nu = 1.
\]

This allows one to prove that this digraph is a disjoint union of cycles, (incl. 1-vertex cycles),

This is quite useful (although not
for us right now),

\textbf{Def.} Let \( n \in \mathbb{N} \) and \( \sigma \in S_n \).

The \textbf{sign of the permutation} \( \sigma \) is the \textit{number} of \textbf{inversions} of \( \sigma \). It is written \( \ell(\sigma) \).

The \textbf{sign of} \( \sigma \) is \( (-1)^{\ell(\sigma)} \). It is denoted by \( (-1)^{\sigma} \) or \( \text{sign } \sigma \) or \( \text{sgn } \sigma \) or \( \varepsilon(\sigma) \).

\textbf{Properties of the sign}:

\begin{itemize}
  \item \( \text{sign } (\sigma) \in \{1, -1\} \),
  \item \( \text{sign } (\text{id}) = 1 \)
  \item \( \text{sign } (\sigma \text{ transposition}) = -1 \).
\end{itemize}
* \[ \text{sign} (\sigma \circ \tau) = \text{sign} \sigma \cdot \text{sign} \tau \quad \forall \sigma, \tau \in S_n. \]

* \[ \text{sign} (\sigma^{-1}) = \text{sign} \sigma. \]

* If the digraph for \( \sigma \) has \( r \) cycles, then \[ \text{sign} \sigma = (-1)^{n-r}. \]

* \[ \text{sign} \sigma = \prod_{1 \leq i < j \leq n} \sigma(i) - \sigma(j). \]

* If you write down the one-line notation \( \alpha_i \) for \( \sigma \), and sort it into increasing order by repeatedly swapping adjacent entries ("bubblesort", or rather a more general version thereof), then \( l(\sigma) \) is the smallest \# of swaps you need. (Actually, it is the exact \# of swaps you need if you don't waste time by swapping pairs that already are increasing.)

For proofs, see references cited in the Introduction of ngraph.pdf, especially [Day 16, Chapter 6, 85].
[Grinbe 16, 84, 1-34, 3], [Conrad].

Having all this out of our way, we can define the determinant.

**Def.** Let $A$ be an $n \times n$-matrix (say, with real entries — not that it matters).

For all $i, j$, let $a_{i,j}$ be the $(i, j)$-th entry of $A$ (i.e., the entry in row $i$ & column $j$).

The determinant $\det A$ of $A$ is defined by

$$\det A = \sum_{\sigma \in S_n} \text{sign } \sigma \cdot \prod_{i=1}^{n} a_{i, \sigma(i)}.$$

This is called the *Heinbrütz* formula. Among many definitions of the determinant, it is the most explicit one.

**Thm. 2 (Vandermonde).** Let $n \in \mathbb{N}$,

let $x_1, x_2, \ldots, x_n$ be $n$ numbers,
Let $V$ be the $n \times n$-matrix whose $(i,j)$-th entry is $x_j^{i-1}$
(therefore,
\[
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}.
\]

Then,
\[
\det V = \prod_{1 \leq i < j \leq n} (x_j - x_i).
\]

(“Vandermonde determinant”, in one of its many forms)

We shall prove this using tournaments.
Proof. ([Ira Gessel, Gessel79])

Let $\mathcal{T}$ be the set of tournaments with vertex set $\{1, 2, \ldots, n\}$. (Thus, $|\mathcal{T}| = 2^{n(n-1)/2}$.)

For each $D \in \mathcal{T}$, define the following:

- For each arc $a = ij$ of $D$, define the weight $\omega(a)$ of $a$ to be
  \[ (1) \left[ i > j \right] x_i x_j \]
  (where we use Iverson bracket notation).

- Define the weight $\omega(D)$ of $D$ to be
  \[ \prod_{\text{arc of } D} \omega(a) = \prod_{i < j} (1) \left[ i > j \right] x_i x_j. \]

Then,

\[ \prod_{1 \leq i < j \leq n} (x_j - x_i) = \sum_{D \in \mathcal{T}} \omega(D), \]
because expanding the left hand side results in a sum of lots and lots of products, each of which corresponds to choosing either the $x_j$ or the $-x_j$ addend from each factor $x_i - x_j$, which we can encode by a tournament $\mathcal{D} \subseteq \mathcal{F}$. (Namely: if we choose the $x_j$ addend, then we let the tournament have an arc $ij$; otherwise, let it have an arc $ji$.)

Hence, it suffices to show that

$$\text{det } V = \sum_{\mathcal{D} \subseteq \mathcal{F}} \omega(D).$$

To do so, we study the number of 3-cycles in a tournament.

\textbf{Key:} For each $k \in \mathbb{N}$, let

$$\omega_k = \sum_{\mathcal{D} \subseteq \mathcal{F}} \omega(D),$$

where $D$ has exactly $k$ 3-cycles.

Then $\sum_{\mathcal{D} \subseteq \mathcal{F}} \omega(D) = \omega_0 + \omega_1 + \omega_2 + \ldots$

(this infinite sum is well-defined, since
\( w_k = 0 \) for any large enough \( k \). Hence, it remains to prove

\[(4) \quad \det V = w_0 + w_1 + w_2 + \ldots.\]

(Note that as the way we defined 3-cycles, the \# of 3-cycles in a tournament \( D \) is always a multiple of 3, since each 3-cycle \((u,v,w)\) yields two others \((w,u,v)\) and \((u,w,v)\), and we don't equate them. But that's not a problem.)

Let us first study the tournaments without 3-cycles. I claim that they correspond to permutations:

**Def.** Let \( \sigma \in S_n \). Then, define a tournament \( T_\sigma \in \mathcal{T} \) by as follows:

- its arcs should be \((\sigma(i), \sigma(j))\) for \( 1 \leq i < j \leq n \).

**lem. 3.** (a) The tournaments \( T_\sigma \in \mathcal{T} \) having 0 3-cycles are precisely those of the form \( T_\sigma \) for \( \sigma \in S_n \).

(b) Any \( \sigma \in S_n \) can be reconstructed uniquely from \( T_\sigma \).
(c) Any \( \sigma \in S_n \) satisfies
\[
\omega(T_\sigma) = \text{sign} \ \sigma \cdot \prod_{i=1}^{n-2} \sigma(i+1).
\]

Proof of Lem. 3. (a) Clearly, \( T_\sigma \) has no 3-cycles
(because if we had a 3-cycle, then we could write it as
\((\sigma(u), \sigma(v), \sigma(w))\), and thus we would have \( u < v, v < w \)
and \( w < u \Rightarrow 3 \)).

Conversely: Let \( D \) be a tournament with no 3-cycles,
we must find a \( \sigma \in S_n \) such that \( D = T_\sigma \).

We know from Lecture 7 that every tournament has a Hamiltonian path.
Thus, \( D \) has one, let it be \((\sigma(1), \sigma(2), \ldots, \sigma(n))\).
Thus, \( \sigma \in S_n \). We claim that \( D = T_\sigma \). Why?

We know that
\[(\sigma(i), \sigma(i+1)) \text{ is an arc of } D \forall i \]
(since \((\sigma(1), \sigma(2), \ldots, \sigma(n))\) is a Hamiltonian path.)
path). Thus,

\((o(i), o(i+2))\) is an arc of \(D\) \(Vi\)

(because otherwise, \((o(i+2), o(i))\) would be an arc of \(D\) instead, but then,
\((o(i), o(i+1), o(i+2))\) would be a 3-cycle, which we knew \(D\) has not). Thus,

\((o(i), o(i+3))\) is an arc of \(D\) \(Vi\)

(because otherwise, \((o(i+3), o(i))\) would be an arc of \(D\) instead, but then,
\((o(i), o(i+2), o(i+3))\) would be a 3-cycle, which we knew \(D\) has not).

Continuing the same logic, we find that

\((o(i), o(i+k))\) is an arc of \(D\) \(Vi\) \(\forall k>0\).

In other words, \((o(i), o(j))\) is an arc of \(D\) \(Vi\) if,

In other words, \(D\) has all the arcs of \(T_0\). And no further arcs, since \(D\) is a tournament and cannot have more than 1 arc between two given vertices.

So \(D=T_0\). This completes the proof of (c).
(b) I claim that \((e(1), e(2), \ldots, e(n))\)

is the only Hamiltonian path of \(T_o\).

Once this is proven, reconstruction of \(e\) from \(T_o\) will be trivial.

Thus let \((\tau(1), \tau(2), \ldots, \tau(n))\)

be any Hamiltonian path of \(T_o\).

We must prove \(\tau = e\).

Clearly, \(\tau \in S_n\). If \(\tau \neq e\), then

\(\sigma^{-1} \circ \tau \neq id\), thus \(\exists k \in \{1, 2, \ldots, n-1\}\)

for which \((\sigma^{-1} \circ \tau)(k) > (\sigma^{-1} \circ \tau)(k+1)\).

Consider this \(k\). Then,

\((\tau(k), \tau(k+1))\) is an arc of \(T_o\).

By the definition of \(T_o\), since

\((\sigma^{-1}(\tau(k)), \sigma^{-1}(\tau(k+1)))\) is a Hamiltonian path,

but also

\((\tau(k+1), \tau(k))\) is an arc of \(T_o\).

(since \((\tau(1), \tau(2), \ldots, \tau(n))\) is a Hamiltonian path), but also

\((\nu(\tau(k+1))) = (\sigma^{-1} \circ \tau)(k+1)\)

\((\nu^{-1} \circ \tau)(k) = (\sigma^{-1} \circ \tau)(k)\).
and by the construction of $T_0$.
This is a contradiction, since $T_0$ is a tournament and cannot have 2 arcs between any given 2 vertices.
So $\tau \neq 0$ cannot happen. Hence, $\tau = 0$.

(Alternatively, look at $T_0$:

This is how $T_0$ looks like.

All arcs go from left to right, so a Hamiltonian path must traverse $o(1), o(2), \ldots, o(n)$ in this order.

(c) $\omega(T_0) = \prod_{\alpha \text{ is an arc of } T_0} \omega(\alpha)$

\[
= \prod_{1 \leq i < j \leq n} \omega(o(i), o(j))
\]
(by the definition of $T_0$)

$$
= \prod_{1 \leq i < j \leq n} \left( (-1)^{[0(i) > 0(j)]} x_{0(i)} \right)
$$

(by the definition of weights)

$$
= \left( \prod_{1 \leq i < j \leq n} (-1)^{[0(i) > 0(j)]} \right) \prod_{1 \leq i \leq n} x_{0(i)}
$$

$$
= (-1)^{\sum_{1 \leq i < j \leq n} [0(i) > 0(j)]}
$$

# of inversions of $\sigma$

$$
= (-1)^{\ell(\sigma)} = \text{sign } \sigma
$$

$$
= \prod_{i=1}^{n} x_{0(i)}^{i-1}
$$

$$
= \text{sign } \sigma \cdot \prod_{i=1}^{n} x_{0(i)}^{i-1}
$$

Thus, Lemma 3 (c) is proven. $\Box$
Cor. 4, \quad \det V = \omega_0.

Proof of Cor. 4. The definition of a determinant yields

\[
\det V = \sum_{\sigma \in S_n} \text{sgn } \sigma \cdot \prod_{i=1}^{n} x_{\sigma(i)}^{i-1}
\]

\[
= \omega(T_0)
\]

(by Lemma 3(c))

\[
= \sum_{\sigma \in S_n} \omega(T_0) = \sum_{D \in \mathcal{P} \mid \text{ D has exactly 0 3-cycles}} \omega(D)
\]

(by Lemma 3 (a) & (b))

\[
= \omega_0.
\]

Recall that our goal is to prove (4). In light of Cor. 4, it suffices to show that \( \omega_k = 0 \) \( \forall k > 0 \).
Thus, let us fix $k > 0$. We want to prove $w_k = 0$.

We shall prove something slightly stronger:

**Lem. 5.** Let $(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$.

Then $\sum_{D \in \mathcal{T}} \text{sign } D = 0$, where $\text{sign } D = \prod_{(i, j) \text{ is an arc of } D} (-1)^{\ell - 1}$, for $\ell = 1, 2, 3$.

**Proof of Lem. 5.** A **flippy pair** will mean a pair $(D, \alpha)$ where $D \in \mathcal{T}$ is a tournament with exactly $k$ 3-cycles and having $\deg_D^-(i) = d_i$, for $i = 1, \ldots, n$, and where $\alpha$ is a 3-cycle of $D$.

If $(D, \alpha)$ is a flippy pair, then flip $(D, \alpha)$ shall mean the pair
Let \((D, \sigma)\) be a 3-cycle \(\alpha\). Let \(D'\) be the tournament obtained from \(D\) by reorienting the arcs \(w_1, w_2, w_3\).

Let \(\alpha'\) be the 3-cycle \((u, w_3, v)\) of \(D'\).

To see that this \((D', \alpha')\) is indeed a flippy pair, we must observe that \(D'\) has exactly \(k\) 3-cycles (by Prop. 1) and satisfies \(\deg_D^-(i) = \deg_{D'}^-(i)\) for all \(i\) (since the \# indegrees do not change from \(D\) to \(D'\)). Moreover, \(\text{flip}(D', \alpha') = (D, \alpha)\)

and \(\text{sign}(D') = \text{sign} D\) (since in the definition of \(\text{sign} D\), precisely 3 factors in the product change their sign when we pass from \(D\) to \(D'\)).

Thus, we have two mutually inverse bijections.
\{ \text{flippy pairs } (\phi, \alpha) \text{ with } \text{sign } D = 1 \} \\
\rightarrow \{ \text{flippy pairs } (\phi, \alpha) \text{ with } \text{sign } D = -1 \}, \\
(\phi, \alpha) \rightarrow \text{flip}(\phi, \alpha)

and

\{ \text{flippy pairs } (\phi, \alpha) \text{ with } \text{sign } D = -1 \} \\
\rightarrow \{ \text{flippy pairs } (\phi, \alpha) \text{ with } \text{sign } D = 1 \}, \\
(\phi, \alpha) \rightarrow \text{flip}(\phi, \alpha).

Hence, the sets

\{ \text{flippy pairs } (\phi, \alpha) \text{ with } \text{sign } D = -1 \}

and

\{ \text{flippy pairs } (\phi, \alpha) \text{ with } \text{sign } D = 1 \}

have the same size. Thus,

\[ \sum_{(\phi, \alpha) \text{ is a flippy pair}} \text{sign } D = (a \text{ sum of several } 1s \text{ and equally many } -1s) = 0. \]
But the left hand side of this equality is

$$k \cdot \sum_{\text{D} \in \mathcal{T};} \text{sign} \, \text{D}$$

\text{D has exactly k 3-cycles;}
\text{deg}_D(v_i) = d_i \cdot V_i

(because each \(D \in \mathcal{T}\) having exactly \(k\) 3-cycles and satisfying \(\text{deg}_D(v_i) = d_i \cdot V_i\)

is part of precisely \(k\) flippy pairs – namely, \(1\) for each of its \(k\) 3-cycles).

Hence, dividing by \(k\), we obtain the claim of Lem. 5.

Now,

$$w_k = \sum_{\text{D} \in \mathcal{T};} \omega(D)$$

\text{D has exactly k 3-cycles}

$$= \sum_{(d_2, d_3, \cdots, d_n) \in \mathbb{N}^n} \sum_{\text{D} \in \mathcal{T};} \omega(D)$$

\text{D has exactly k 3-cycles;}
\text{deg}_D(v_i) = d_i \cdot V_i$$
\[
\sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n} \sum_{\text{De } \mathcal{T} \text{ has exactly } k \text{ 3-cycles}, \text{ deg}_D(i) = d_i \forall i} (\text{sign } D) \prod_{j=1}^{n} x_{i_j}^{d_{i_j}}
\]

(because for each De \( \mathcal{T} \) with \( \text{deg}_D(i) = d_i \forall i \), we have
\[
\omega(D) = \prod_{\text{y is an arc of } D} \left( \prod_{i \neq j} (-1)^{x_i x_j} \right)
\]

\[
= \left( \prod_{\text{y is an arc of } D} (-1)^{x_i x_j} \right) \left( \prod_{\text{y is an arc of } D} x_{i_j} \right) = \text{sign } D \prod_{i=1}^{n} \prod_{j \neq i}^{n} x_{i_j}
\]

\[
= (\text{sign } D) \cdot \prod_{j=1}^{n} \prod_{\text{i is even, } i \neq j}^{n} x_{i_j} = x_{i_j}^{\text{even}}
\]

\[
= (\text{sign } D) \cdot \prod_{j=1}^{n} x_{i_j}^{d_{i_j}}
\]
\[
\sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n} \left( \sum_{D \in \mathcal{D}} \text{Sign } D \right) \prod_{f=1}^{n} x_f^{d_f}
\]

where

- \(D \in \mathcal{D}\) has exactly
  - 3-cycles,
  - \(\deg_D(i) = d_i \forall i\)

\(= 0\) (by Lem. 5)

\[\sum_{(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n} \left( \sum_{D \in \mathcal{D}} \text{Sign } D \right) \prod_{f=1}^{n} x_f^{d_f} = 0,\] and so we're done. \(\square\)