Exercise 1

(a) It is sufficient to prove that there are at least two triangle-or-anti-triangles for $|V(G)| = 6$, because any graph with more than 6 vertices can be viewed as a graph with 6 vertices by ignoring the extra vertices and their edges. For example,

According to Proposition 2.4.1, $G$ already has at least one triangle-or-anti-triangle. Let this triangle-or-anti-triangle be a triangle with vertices $a$, $b$, and $c$. Consider the other three vertices $x$, $y$, and $z$. If $xy$, $yz$, and $zx$ are all edges, then $xyz$ is a triangle and we’re done. So, assume that at least one edge, $xy$ is a non-edge. Either two out of the three $xa$, $xb$, $xc$ are edges, or two out of the three are non-edges. If two out of the three are edges, then we’re done. So assume that two of the three edges are non-edges. Similarly, assume that two out of the three edges $ya$, $yb$, and $yc$ are non-edges. By the pigeonhole principal, either $xa$ and $ya$, $xb$ and $yb$, or $xc$ and $yc$ are both non-edges, which forms an anti-triangle with $xy$.

An analogous argument for when $abc$ is an anti-triangle comes to a similar conclusion, replacing edge with non-edge and etc.

(b) When $m = 0$, $|V(G)| = 6$, so by Proposition 2.4.1, $G$ has one triangle-or-anti-triangle.

Assume there are $k + 1$ triangle-or-anti-triangles when $m = k$.

Suppose there is a graph $H$ where $|V(H)| = (k + 1) + 6$. According to Proposition 2.4.1, this graph must have at least one triangle-or-anti-triangle with vertices $a$, $b$, and $c$. Ignore vertex $a$, so that you’re viewing a graph with $k$ vertices. This graph must have $k + 1$ triangle-or-anti-triangles, none of which are $abc$ because we have ignored $a$. Thus, $abc$ plus these $k + 1$ triangle-or-anti-triangles means that $H$ has $(k + 1) + 1$ triangle-or-anti-triangles total.

Thus, by induction, $G$ has at least $m + 1$ triangle-or-anti-triangles.

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1 slight updates by DG, 2017 May 05
**Exercise 2.**

A graph $G = (V, E)$ has $|V(G)| = n$. The most number of edges $G$ could have is $\binom{n}{2}$, or $\frac{n(n-1)}{2}$. If $|E(G)| < \frac{n(n-2)}{4}$, the graph $G$ has over $\frac{n(n-1)}{2} - \frac{n(n-2)}{4} = \frac{2n^2 - 2n}{4} - \frac{n^2 - 2n}{4} = \frac{n^2}{4}$ non-edges. Theorem 2.5.10 states that a simple graph with more than $\frac{n^2}{4}$ edges must have $a, b, c \in G$ such that $ab$, $bc$, and $ca$ are edges. As such, it should still be true if we consider those $\frac{n^2}{4}$ edges as non-edges, and $ab$, $bc$, and $ca$ as non-edges. Since we have more than $\frac{n^2}{4}$ non-edges, there must be $a, b, c \in G$ such that $ab$, $bc$, $cb$ are not edges.

**Exercise 3.**

By definition, $w$ in $G$ has vertices $v_0, v_1, v_2, \ldots, v_k$ such that for $0 \leq i, j \leq k$, $v_i \neq v_j$. Let $0 \leq n, m \leq k$, and $w$ be $(v_0, v_1, \ldots, v_n, v_{n+1}, \ldots, v_m, v_{m+1}, \ldots, v_k)$. If $w$ has a repeated edge, then $\{v_n, v_{n+1}\} = \{v_m, v_{m+1}\}$ for some $n, m$. However, $v_i \neq v_j$, so $v_n \neq v_m \neq v_{m+1}$. Thus, each edge must be distinct.

**Exercise 4.**

A cycle is a walk $w$ with vertices $(v_0, v_1, \ldots, v_k)$ and $v_k = v_0$ but for $0 < i, j < k$, $v_i \neq v_j$. Label $C_{3n}$ as follows.

$C_{3n}$ has edges $v_1v_{12}, v_{12}v_{13}, \ldots, v_{n2}v_{n3}, v_{n3}v_{11}$.

![Image of a cycle graph]

Say we’re constructing a dominating set $U$. Each $v \in C_{3n}$ has degree exactly 2. So if we choose a vertex $v$ to be part of $U$, its two neighbors satisfy the conditions that it has a neighbor in $U$.

So at most, choosing one vertex accounts for 3 vertices: $v$ itself, which is in $U$, and its 2 neighbors.

So the minimum number of vertices needed to dominate $V(C_{3n})$ is $|V(C_{3n})|/3 = 3n/3 = n$.

For example, choose all the $v_{k2}$ vertices to form $U$.

**Exercise 5.**

**Proposition 0.2**

(i) If $A$ and $B$ are true, $[A] = 1$, $[B] = 1$. If $A$ and $B$ are false, $[A] = 0$, $[B] = 0$.

$\Rightarrow [A] = [B]$.

(ii) If $A$ is true, then $[A] = 1$, $[\neg A] = 0$, and $1 - [A] = 0$ If $A$ is false, then $[A] = 0$, $[\neg A] = 1$, and $1 - [A] = 1$.

Thus, $[\neg A] = 1 - [A]$.

(iii) If both $A$ and $B$ are true, then $[A] = 1$, $[B] = 1$, $[A \land B] = 1$, and $[A][B] = 1$.

If both $A$ and $B$ are false, then $[A] = 0$, $[B] = 0$, $[A \land B] = 0$, and $[A][B] = 0$.

If $A$ is true and $B$ is false, then $[A] = 1$, $[B] = 0$, $[A \land B] = 0$, and $[A][B] = 0$.

If $A$ is false and $B$ is true, then $[A] = 0$, $[B] = 1$, $[A \land B] = 0$, and $[A][B] = 0$.

Thus, $[A \land B] = [A][B]$.

(iv) If both $A$ and $B$ are true, then $[A] = 1$, $[B] = 1$, $[A \lor B] = 1$, and $[A] + [B] - [A][B] = 1$.

If both $A$ and $B$ are false, then $[A] = 0$, $[B] = 0$, $[A \lor B] = 0$, and $[A] + [B] - [A][B] = 0$.

If $A$ is true and $B$ is false, then $[A] = 1$, $[B] = 0$, $[A \lor B] = 1$, and $[A] + [B] - [A][B] = 1$.

If $A$ is false and $B$ is true, then $[A] = 0$, $[B] = 1$, $[A \lor B] = 1$, and $[A] + [B] - [A][B] = 1$.

Thus, $[A \lor B] = [A][B] + [B] - [A][B]$.  

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Proposition 0.3

(i) |Q| is equivalent to the number of elements of P are in Q. Thus, each p is in Q gives the statement p ∈ Q is true, with a truth value of 1. if p is not in Q, then p ∈ Q is 0. So the the sum \( \sum_{p \in P} |p \in Q| \) accurately gives |Q|.

(ii) \( \sum_{p \in P} [p \in Q]a_p = (0)a_p = 0 \) if p \( \notin \) Q. Thus, if we rewrite the sum by skipping over all p \( \notin \) Q, we get \( \sum_{p \in Q} [p \in Q]a_p = \sum a_p \).

(iii) \( [p = q] = 0 \) when p \( \neq \) q. Thus, \( [p = q]a_p = 0 \) for all p \( \in \) P \( \neq \) q. When p = q, \( [p = q] = 1 \), so \( [p = q]a_q = a_q \).

(c) By Definition 2.5.1 \( \text{deg } v = |\{u \in V \mid uv \in E\}| \).

[d] According to result (c),

\[
\sum_{u \in V} \sum_{v \in V} |uv \in E| = \sum_{u \in V} \text{deg } u
\]

Think of \( \text{deg } u \) as the number of endpoints u act as. That is, if \( \text{deg } u = 5 \), then u is the endpoint of 5 edges. Then the sum of all the degrees of u are equal to twice the number of edges because each edge has two endpoints, implying that the sum of all the degrees of u double counts the edges. So,

\[2|E| = \sum_{u \in V} \text{deg } u = \sum_{u \in V} \sum_{v \in V} [uv \in E].\]

Exercise 6.

A k-path-dominating subset U of V(G) can be thought of as a 1-path-dominating subset of V(G) by drawing an edge between all u, v \( \in \) V such that the smallest path w from u \( \rightarrow \) v has length k. Since 1-path-dominating subsets have an odd number of subsets according to Brouwer’s theorem, so do k-path-dominating subsets which can be thought of as 1-path-dominating subsets!

Exercise 7.

First let’s show that Statement 1 \( \Rightarrow \) Statement 2.

The definition of a graph G being connected is that for each u, v \( \in \) V, there exists a path from u \( \rightarrow \) v. Suppose V was divided into nonempty subsets A, B such that for all a \( \in \) A and b \( \in \) B, ab \( \notin \) E. That means there is no path from any a to any b, which is a contradiction because there is a path a \( \rightarrow \) b for all a and b because G is connected.

Next let’s show that Statement 2 \( \Rightarrow \) Statement 1. Let the vertices of V be called \( v_0, v_1, v_2, ..., v_k \). Choose \( v_0 \) to be A, and \( V \setminus v_0 \) be B. According to Statement 2, there is some \( v_m \in B \) such that \( v_0 \) is adjacent to \( v_m \).

Now choose \( v_0 \) and \( v_m \) to be A, and B to be \( V \setminus A \). According to Statement 2, there is some \( v_m \in B \) such that \( v_0 \) or \( v_m \) is adjacent to \( v_m \). Say \( v_m \) is adjacent to \( v_m \). Then there is also a path \( v_m \rightarrow v_m \), \( (v_m, v_0, v_m) \). This makes \( v_m, v_m, v_0 \) connected.

Now choose \( v_0, v_m \) and \( v_m \) to be A, and B to be \( V \setminus A \). According to Statement 2, there is some \( v_p \) such that \( v_p v_X \in E \) (where \( X = m,n,0 \)). Say \( v_p \) is an edge. \( v_p \) is connected to the rest of A. A is connected which means that there is a path between all \( a_1, a_2 \in A \). Say \( v_p \) is connected to \( v_m \) by path \( (v_0,...,v_m) \) Thus the edge \( v_p v_0 \) concatenated with \( v_0 \rightarrow v_m \) in A, makes a path \( (v_p, v_0, v_0, v_m) \).

And so on and so forth until B has only one element, which shows that all of G is connected.

Sorry, I think that was really confusing.

Exercise 8.

Suppose that G is not connected, that is, there is some \( u_0, v_0 \in V \) such that \( u_0, v_0 \notin G \). This means that the path \( (u_0, ..., v_0) \) \( u_0, v_0 \) exists in H.

The other vertices a besides \( u_0 \) and \( v_0 \) may have one of two properties: (i) There exists a path \( a \rightarrow v_0 \) (a,...,v_0) in G, or (ii) there isn’t a path \( a \rightarrow v_0 \) in G.

If \( a \rightarrow v_0 \) exists in G, then \( u_0 \rightarrow a \) does not. Otherwise, a walk \( (u_0, ..., a, ..., v_0) \) could be constructed from \( u_0 \rightarrow v_0 \) (which implies there is a path), which we have said cannot exist. As such, \( u_0 \rightarrow a \) exists in H.
If \( a \to v_0 \) does not exist in \( G \), then \( a \to v_0 \ (a,...v_0) \) must exist in \( H \). Thus, \( (u_0,...v_0...a) \) is a walk in \( H \), which means that there is a path from \( u_0 \to a \) in \( H \).

If there is a path \( u_0 \to v \) for all \( v \in V \), then there is a walk between any two \( v \) because \( v_i \to u_0(v_i,...u_0) \) and \( u_0 \to v_j(u_0...v_j) \) can be used to form a walk \( (v_i...u_0...v_j) \Rightarrow \) there is a path from \( v_i \to v_j \).

**Exercise 9.**

Let Statement 1 not hold for \( G \), i.e. not all vertices \( u, v \in G \) have a path \( w \) from \( u \to v \) such that \( |w| \leq 3 \).

What could the graph \( \overline{G} \) look like?

For every \( u, v \in G \) there are several cases.

One, \( u \) and \( v \) are not adjacent in \( G \). Thus, \( uv \notin G \) and \( uv \in \overline{G} \). So path \( \overline{w}(u,v) \in \overline{G} \) exists, and \( |w| = 1 \leq 2 \).

Two, \( u \) and \( v \) are adjacent in \( G \).

Let \( A = \{ a \mid au \in G \land a \neq v \} \) and \( B = \{ b \mid bv \in G \land b \notin A \land b \neq u \} \).

All \( a \in A \) are connected by a a path \( (a, u, v, b) \) of length 3 to all \( b \in B \), by a path \( (a, u, v) \) of length 2 to \( v \), and by a path \( (a, u) \) of length 1 to \( u \).

Similarly, all \( b \in B \) are connected by a a path \( (b, v, u, a) \) of length 3 to all \( a \in A \), by a path \( (b, v, u) \) of length 2 to \( u \), and by a path \( (b, v) \) of length 1 to \( v \).

Thus, there must be a vertex \( w \) s.t. \( wu, uv \notin G \), otherwise Statement 1 would hold, which imples \( wu, uv \in \overline{G} \), which creates a path \( w' (u, w, v) \) where \( |w'| \leq 2 \).

Thus, \( \forall u, v \in V \), \( \exists \) a path \( w \) in \( \overline{G} \) s.t. \( |w| \leq 2 \).

Let Statement 2 not hold for \( \overline{G} \).

For every \( u, v \in \overline{G} \) there are several cases.

One, \( u \) and \( v \) are not adjacent in \( G \). Thus, \( uv \notin G \) and \( uv \in G \). So path \( \overline{w}(u,v) \in G \) exists, and \( |w| = 1 \leq 3 \).

Two, \( u \) and \( v \) are adjacent in \( G \).

Let \( C = \{ c \mid cw, cw \in \overline{G} \} \) (which could be empty) and \( A = \{ a \mid au \in \overline{G} \land a \neq v \land a \notin C \} \) and \( B = \{ b \mid bv \in \overline{G} \land b \notin A \land b \neq u \land b \notin C \} \).

All \( a \in A \) are connected by a a path \( (a, u, v, b) \) of length 3 to all \( b \in B \), by a path \( (a, u, v) \) of length 2 to \( v \), by a path \( (a, u, c) \) of length 2 to \( c \) (if \( C \neq \emptyset \)) and by a path \( (a, u) \) of length 1 to \( u \).

Similarly, all \( b \in B \) are connected by a a path \( (b, v, u, a) \) of length 3 to all \( a \in A \), by a path \( (b, v, u) \) of length 2 to \( u \), by a path \( (a, v, c) \) of length 2 to \( c \) (if \( C \neq \emptyset \)) and by a path \( (b, v) \) of length 1 to \( v \).

Thus, there must be vertices \( a_1 \in A \) and \( b_1 \in B \) such that \( ab \notin \overline{G} \). If not, then there would be a path \( (a, b) \) of length 1 for all \( a \) and \( b \).

As such, \( ab, va, ub \) are not in \( G \Rightarrow ab, va, ub \in G \). So there is a path \( w' (u, a, b, v) \) where \( |w'| \leq 3 \).

Thus, \( \forall u, v \in V \), \( \exists \) a path \( w \) in \( G \) s.t. \( |w| \leq 3 \).