Exercise 1. Let $G$ be a simple graph. A triangle in $G$ means a set $\{a,b,c\}$ of three distinct vertices $a,b,$ and $c$ of $G$ such that $ab,bc,$ and $ca$ are edges of $G$. An antitriangle in $G$ means a set $\{a,b,c\}$ of three distinct vertices $a,b,$ and $c$ of $G$ such that none of $ab,bc,$ and $ca$ is an edge of $G$. A triangle-or-antitriangle in $G$ is a set that is either a triangle or an antitriangle.

(a) Assume that $|V(G)| \geq 6$. Prove that $G$ has at least two triangle-or-antitriangles.

Proof: To begin, assume, without loss of generality, that $|V(G)| = 6$, and note that there are $\binom{6}{3} = 20$ unordered triples of vertices in $G$. We wish to count the number of ordered triples $(a,b,c)$ such that $ab \in E(G)$ and $bc \notin E(G)$. Observe that each vertex $v$ has $\deg v = 0, 1, 2, 3, 4,$ or $5$. If $\deg v$ is $0$ or $5$, then $v$ is the second vertex in $0$ such triples. If $\deg v$ is $1$ or $4$, then $v$ is the second vertex in $4$ such triples. If $\deg v$ is $2$ or $3$, then $v$ is the second vertex in $6$ such triples. Thus, there are at most $36$ such ordered triples. Note that each unordered triple of vertices that does not form a triangle-or-antitriangle corresponds to two such ordered triples, so at most $18$ unordered triples of vertices do not form triangle-or-antitriangles, so there are at least two triangle-or-antitriangles in $G$.

(b) Assume that $|V(G)| = m + 6$ for some $m \in \mathbb{N}$. Prove that $G$ has at least $m + 1$ triangle-or-antitriangles.

Proof: Let $V(G) = \{v_1,v_2,...,v_6,u_1,...,u_m\}$. Now, take $\{v_1,...,v_6\}$. By the above, we can find at least one triangle-or-antitriangle among these six vertices. Suppose, without loss of generality, that this triangle-or-antitriangle includes $v_1$. Then, we take the vertices $\{v_2,...,v_6,u_1\}$. Again, there is at least one triangle-or-antitriangle among these six vertices, and since $v_1$ is not included, this triangle-or-antitriangle must be distinct from the one found previously. By repeating this process of finding a triangle-or-antitriangle, removing one of its vertices, and replacing it by one of the $u_k$, we find at least one new triangle-or-antitriangle for each of $\{u_1,...,u_m\}$, plus one in which none of the $u_k$ were used.
Thus, we find \( m + 1 \) triangle-or-antitriangles. ■

**Exercise 2.** Let \( G \) be a simple graph. Let \( n = |V(G)| \) be the number of vertices of \( G \). Assume that \( |E(G)| < \frac{n(n-2)}{4} \). Prove that there exist three distinct vertices \( a, b, \) and \( c \) of \( G \) such that none of \( ab, bc, \) and \( ca \) are edges of \( G \).

**Proof:** Suppose that for all distinct \( a, b, c \in V(G) \), at least one of \( ab, bc, ca \) \( \in E(G) \). Then the complement \( \overline{G} \) of \( G \) is a graph with no triangles. We know \( |E(\overline{G})| = \frac{n(n-1)}{2} - |E(G)| \), so that \( |E(G)| = \frac{n(n-1)}{2} - |E(\overline{G})| \). But by Mantel’s theorem we know \( |E(G)| \leq \frac{n^2}{4} \) (since \( \overline{G} \) has no triangles), and thus \( |E(G)| = \frac{n(n-1)}{2} - |E(\overline{G})| \geq \frac{n(n-1)}{2} - \frac{n^2}{4} = \frac{n(n-2)}{4} \). Thus, if \( |E(G)| < \frac{n(n-2)}{4} \), then there exist \( a, b, c \in V(G) \) such that \( ab, bc, ca \notin E(G) \). ■

**Exercise 3.** Let \( G \) be a simple graph. Let \( w \) be a path in \( G \). Prove that the edges of \( w \) are distinct.

**Proof:** Let \( \{v_0, v_1, ..., v_k\} \) be the vertices of \( w \). Since \( w \) is a path, its vertices are distinct. Now, the edges of \( w \) are \( \{v_i, v_{i+1} \mid 0 \leq i \leq k-1\} \). Note that two edges \( pq \) and \( rs \) are the same only when \( p = r \) and \( q = s \) or \( p = s \) and \( q = r \), i.e. if they connect the same pair of vertices. Now, each vertex \( v_i \) in \( w \) is connected to at most two edges in \( w \), but these two edges must be distinct because the vertices \( v_{i-1} \) and \( v_{i+1} \) are distinct. ■

**Exercise 4.** Let \( n \in \mathbb{N} \). What is the smallest possible size of a dominating set of the cycle graph \( C_{3n} \)?

The smallest possible size of a dominating set of \( C_{3n} \) is \( n \).

**Proof:** Let the vertices of \( C_{3n} \) be \( \{v_1, v_2, ..., v_{3n}\} \). We observe that since each vertex in \( C_{3n} \) has 2 neighbors, no three consecutive vertices can be excluded from a dominating set. Then, if we pick vertices such that every third vertex is in our dominating set, we have the set \( \{v_3, v_6, ..., v_{3n}\} \), which has \( n \) vertices. ■

**Exercise 5. Proposition 0.2 (a)** If \( A \) and \( B \) are two equivalent logical statements, then \( |A| = |B| \).

(b) If \( A \) is any logical statement, then \( |\text{not} \ A| = 1 - |A| \).

(c) If \( A \) and \( B \) are two logical statements, then \( |A \land B| = |A||B| \).
If \( A \) and \( B \) are two logical statements, then \([A \lor B] = [A] + [B] - [A][B]\).

**Proposition 0.3** Let \( P \) be a finite set. Let \( Q \) be a subset of \( P \).

(a) Then, 

\[
|Q| = \sum_{p \in P} [p \in Q].
\]

(b) For each \( p \in P \), let \( a_p \) be a number. Then, 

\[
\sum_{p \in P} [p \in Q] a_p = \sum_{p \in Q} a_p.
\]

(c) For each \( p \in P \), let \( a_p \) be a number. Let \( q \in P \). Then, 

\[
\sum_{p \in P} [p = q] a_p = a_q.
\]

(a) Prove Proposition 0.2.

**Proof of (a):** If \( A \) and \( B \) are equivalent logical statements, then \( A \) is true if and only if \( B \) is true. Thus, \([A] = [B]\). ■

Proof of (b): If \([A] = 1\), then \([\neg A] = 0 = 1 - 1\). If \([A] = 0\), then \([\neg A] = 1 = 1 - 0\). ■

Proof of (c): If \([A] = [B] = 1\), \([A \land B] = 1 = [A][B]\). If \([A] = 0\) or \([B] = 0\), then \([A \land B] = 0 = [A][B]\). ■

Proof of (d): If \([A] = [B] = 0\), \([A \lor B] = 0 = [A] + [B] - [A][B]\). If \([A] = 1\) or \([B] = 1\), then \([A \lor B] = 1 = [A] + [B] - [A][B]\). ■

(b) Prove Proposition 0.3.

**Proof of (a):**

\[
\sum_{p \in P} [p \in Q] = \sum_{p \in Q} [p \in Q] + \sum_{p \in P \setminus Q} [p \in Q] = \sum_{p \in Q} 1 + \sum_{p \in P \setminus Q} 0 = \sum_{p \in Q} 1 = |Q|. ■
\]

**Proof of (b):**

\[
\sum_{p \in P} [p \in Q] a_p = \sum_{p \in Q} [p \in Q] a_p + \sum_{p \in P \setminus Q} [p \in Q] a_p = \sum_{p \in Q} a_p. ■
\]
Proof of (c):
\[
\sum_{p \in P} [p = q]a_p = \sum_{p \neq q} 0 + \sum_{p = q} a_p = a_q. \blacksquare
\]

(c) Now, let \(G\) be a simple graph. Prove that
\[
\deg v = \sum_{u \in V(G)} [uv \in E(G)]
\]
for each vertex \(v\) of \(G\).

**Proof:** Let \(A \subset V(G)\) be the set of neighbors of \(v\). Then,
\[
\deg v = |A| = \sum_{u \in V(G)} [u \in A].
\]
Now, \(u \in A\) is equivalent to \(uv \in E(G)\), so
\[
\sum_{u \in V(G)} [u \in A] = \sum_{u \in V(G)} [uv \in E(G)]. \blacksquare
\]

(d) Prove that
\[
2|E(G)| = \sum_{u \in V(G)} \sum_{v \in V(G)} [uv \in E(G)].
\]

**Proof:**
\[
2|E(G)| = \sum_{v \in V(G)} \deg v = \sum_{v \in V(G)} \sum_{u \in V(G)} [uv \in E(G)]. \blacksquare
\]

**Exercise 6.** Let \(k\) be a positive integer. Let \(G\) be a graph. A subset \(U\) of \(V(G)\) will be called \(k\)-path-dominating if for every \(v \in V(G)\), there exists a path of length \(\leq k\) from \(v\) to some element of \(U\). Prove that the number of all \(k\)-path-dominating subsets of \(V(G)\) is odd.

**Proof:** Consider the case of the 1-path-dominating subsets. As was proven by Brouwer, the number of such subsets is odd in any graph. Now, construct the graph \(G_k\) by adding to \(G\) edges between any two vertices that are connected by a path of length \(\leq k\) in \(G\). Then, a dominating set of \(G_k\) is a \(k\)-path-dominating subset of \(G\), and \(G_k\) must have an odd number of dominating sets. \(\blacksquare\)

**Exercise 7.** Let \(G\) be a simple graph with \(V(G) \neq \emptyset\). Show that the following two statements are equivalent:

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**Statement 1:** The graph $G$ is connected.

**Statement 2:** For every two nonempty subsets $A$ and $B$ of $V(G)$ satisfying $A \cap B = \emptyset$ and $A \cup B = V(G)$, there exist $a \in A$ and $b \in B$ such that $ab \in E(G)$.

**Proof:** First, we prove that Statement 1 implies Statement 2. Assume $G$ is connected. Since $G$ is connected, there exists a path between any pair of vertices $\alpha, \beta \in V(G)$. Let such a path be $(\alpha, v_1, \ldots, v_k, \beta)$. Without loss of generality, suppose $V(G)$ is divided into $A$ and $B$ such that $\alpha \in A$ and $\beta \in B$. Then, since the above path begins with a vertex in $A$ and ends with a vertex in $B$, it must have at least one edge connecting some $a \in A$ and $b \in B$. Now, to show Statement 2 implies statement 1, suppose $G$ is not connected, and that for every two nonempty subsets $A$ and $B$ of $V(G)$ satisfying $A \cap B = \emptyset$ and $A \cup B = V(G)$, there exist $a \in A$ and $b \in B$ such that $ab \in E(G)$.

Since $G$ is not connected, there exist vertices $\alpha$ and $\beta$ such that no path exists from $\alpha$ to $\beta$. Then, we attempt to construct the sets $A$ and $B$ by defining $A = \{ v \in V(G) \mid a \text{ path from } \alpha \text{ to } v \text{ exists}\}$ and $B = V(G) \setminus A$. Now, $\beta \in B$ since no path connects $\alpha$ to $\beta$, but since Statement 2 was assumed, there is an edge connecting $\beta$ to a vertex in $A$, a contradiction. Hence, the above statements are equivalent. ■

**Exercise 8.** Let $V$ be a nonempty finite set. Let $G$ and $H$ be two simple graphs such that $V(G) = V(H) = V$. Assume that for each $u, v \in V$, there exists a path from $u$ to $v$ in $G$ or a path from $u$ to $v$ in $H$. Prove that at least one of the graphs $G$ and $H$ is connected.

**Proof:** Without loss of generality, assume $G$ is not connected. Then, fix a vertex $u \in V$. We can divide $V$ into two nonempty subsets: $A = \{ v \in V(G) \mid a \text{ path from } u \text{ to } v \text{ exists}\}$ and $B = V \setminus A$. Now, since no paths connecting elements of $A$ to elements of $B$ exist in $G$, for all $a \in A, b \in B$ a path from $a$ to $b$ exists in $H$. Then, for any $a_1, a_2 \in A$ (or $b_1, b_2 \in B$), a path from $a_1$ to $a_2$ (or $b_1$ to $b_2$) exists in $H$ since such a path can be constructed from the paths connecting $a_1$ and $a_2$ to any element of $B$ (or $b_1$ and $b_2$ to any element of $A$). Thus, $H$ is connected. ■

**Exercise 9.** Let $G = (V, E)$ be a simple graph. The complement graph $\overline{G}$ of $G$ is defined to be the simple graph $(V, P_2(V) \setminus E)$. (Thus, two vertices $u$ and $v$ are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.) Prove that at least one of the following statements holds:

**Statement 1:** For each $u \in V$ and $v \in V$, there exists a path from $u$ to $v$ in $G$ of length $\leq 3$. 

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**Statement 2:** For each $u \in V$ and $v \in V$, there exists a path from $u$ to $v$ in $\overline{G}$ of length $\leq 2$.

**Proof:** It is sufficient to show that when Statement 1 is false, Statement 2 holds. Thus, assume there exist $u, v \in V$ such that there is no path from $u$ to $v$ in $G$ of length $\leq 3$. Then, $u$ and $v$ are adjacent in $\overline{G}$. We must show that for each pair $(a, b)$ of vertices, there exists a path of length $\leq 2$ between $a$ and $b$ in $\overline{G}$. If the two vertices are not adjacent in $G$, this is trivial (since they are then adjacent in $\overline{G}$). Now, let $a, b \in V$ be a pair of adjacent vertices in $G$. Then, none of the paths\(^1\) $(u, a, b, v), (u, b, a, v), (u, a, v)$, and $(u, b, v)$ exist in $G$ (by our assumption on $u$ and $v$). Hence, we can assume without loss of generality that $ua, ub \notin E(G)$, which implies $ua, ub \in E(\overline{G})$. Hence, the path $(a, u, b)$ of length 2 exists in $\overline{G}$. Therefore, Statement 2 holds whenever Statement 1 is false, so at least one of the two statements holds. ■

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\(^1\)Some of the vertices $u, a, b, v$ might coincide. In this case, you should ignore them. For instance, you should read the path $(u, a, b, v)$ as $(u, a, v)$ in the case when $b = v$. 

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