Exercise 1. Let $G$ and $H$ be two simple graphs. The Cartesian product of $G$ and $H$ is a new simple graph, denoted $G \times H$, which is defined as follows:

- The vertex set $V(G \times H)$ of $G \times H$ is the Cartesian product $V(G) \times V(H)$.
- A vertex $(g, h)$ of $G \times H$ is adjacent to a vertex $(g', h')$ of $G \times H$ if and only if we have
  - either $g = g'$ and $hh' \in E(H)$,
  - or $h = h'$ and $gg' \in E(G)$.

(In particular, exactly one of the two equalities $g = g'$ and $h = h'$ has to hold when $(g, h)$ is adjacent to $(g', h')$.)

(a) Recall the $n$-dimensional cube graph $Q_n$ defined for each $n \in \mathbb{N}$. (Its vertices are $n$-tuples $(a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$, and two such vertices are adjacent if and only if they differ in exactly one entry.) Prove that $Q_n \cong Q_{n-1} \times Q_1$ for each positive integer $n$. (Thus, $Q_n$ can be obtained from $Q_1$ by repeatedly forming Cartesian products; i.e., it is a “Cartesian power” of $Q_1$.)

(b) Assume that each of the graphs $G$ and $H$ has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian path.

(c) Assume that both numbers $|V(G)|$ and $|V(H)|$ are $> 1$, and that at least one of them is even. Assume again that each of the graphs $G$ and $H$ has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian cycle.

Exercise 2. Let $n$ be a positive integer. Recall that $K_n$ denotes the complete graph on $n$ vertices. This is the graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $\mathcal{P}_2(V)$ (so each two distinct vertices are connected).

Find Eulerian circuits for the graphs $K_3$, $K_5$, and $K_7$.

Exercise 3. Let $n$ be a positive integer, and $K$ be a nonempty finite set. Let $k = |K|$. A de Bruijn sequence of order $n$ on $K$ means a list $(c_0, c_1, \ldots, c_{kn-1})$ of $k^n$ elements of $K$ such that

1. for each $n$-tuple $(a_1, a_2, \ldots, a_n) \in K^n$ of elements of $K$, there exists a unique $r \in \{0, 1, \ldots, k^n - 1\}$ such that $(a_1, a_2, \ldots, a_n) = (c_r, c_{r+1}, \ldots, c_{r+n-1})$. 


Here, the indices are understood to be cyclic modulo $k^n$; that is, $c_q$ (for $q \geq k^n$) is defined to be $c_q \mod k^n$, where $q \mod k^n$ denotes the remainder of $q$ modulo $k^n$.

(Note that the condition (1) can be restated as follows: If we write the elements $c_0, c_1, \ldots, c_{k^n-1}$ on a circular necklace (in this order), so that the last element $c_{k^n-1}$ is followed by the first one, then each $n$-tuple of elements of $K$ appears as a string of $n$ consecutive elements on the necklace, and the position at which it appears on the necklace is unique.)

For example, $(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = (1, 1, 2, 2, 3, 3, 1, 3, 2)$ is a de Bruijn sequence of order 2 on the set $\{1, 2, 3\}$, because for each 2-tuple $(a_1, a_2) \in \{1, 2, 3\}^2$, there exists a unique $r \in \{0, 1, \ldots, 8\}$ such that $(a_1, a_2) = (c_r, c_{r+1})$. Namely:

$$(1, 1) = (c_0, c_1); \quad (1, 2) = (c_1, c_2); \quad (1, 3) = (c_6, c_7);$$

$$(2, 1) = (c_8, c_9); \quad (2, 2) = (c_2, c_3); \quad (2, 3) = (c_3, c_4);$$

$$(3, 1) = (c_5, c_6); \quad (3, 2) = (c_7, c_8); \quad (3, 3) = (c_4, c_5).$$

Prove that there exists a de Bruijn sequence of order $n$ on $K$ (no matter what $n$ and $K$ are).

**Hint:** Let $D$ be the digraph with vertex set $K^{n-1}$ and an arc from $(a_1, a_2, \ldots, a_{n-1})$ to $(a_2, a_3, \ldots, a_n)$ for each $(a_1, a_2, \ldots, a_n) \in K^n$ (and no other arcs). Prove that $D$ has an Eulerian circuit.

Recall that the **indegree** of a vertex $v$ of a digraph $D = (V, A)$ is defined to be the number of all arcs $a \in A$ whose target is $v$. This indegree is denoted by $\deg^-(v)$ or by $\deg^-_D(v)$ (whenever the graph $D$ is not clear from the context).

Likewise, the **outdegree** of a vertex $v$ of a digraph $D = (V, A)$ is defined to be the number of all arcs $a \in A$ whose source is $v$. This outdegree is denoted by $\deg^+(v)$ or by $\deg^+_D(v)$ (whenever the graph $D$ is not clear from the context).

**Exercise 4.** Let $D$ be a digraph. Show that $\sum_{v \in V(D)} \deg^-(v) = \sum_{v \in V(D)} \deg^+(v)$.

The next few exercises are about tournaments. A tournament is a loopless digraph $D = (V, A)$ with the following property: For any two distinct vertices $u \in V$ and $v \in V$, precisely one of the two pairs $(u, v)$ and $(v, u)$ belongs to $A$. (In other words, any two distinct vertices are connected by an arc in one direction, but not in both.)

A 3-cycle in a tournament $D = (V, A)$ means a triple $(u, v, w)$ of vertices in $V$ such that all three pairs $(u, v), (v, w)$ and $(w, u)$ belong to $A$.\footnote{A digraph $(V, A)$ is said to be loopless if it has no loops. (A loop means an arc of the form $(v, v)$ for some $v \in V$.)}
Exercise 5. Let \( D = (V, A) \) be a tournament. Set \( n = |V| \) and \( m = \sum_{v \in V} \left( \frac{\deg^- (v)}{2} \right) \).

(a) Show that \( m = \sum_{v \in V} \left( \frac{\deg^+ (v)}{2} \right) \).

(b) Show that the number of 3-cycles in \( D \) is \( 3 \left( \binom{n}{3} - m \right) \).

Exercise 6. If a tournament \( D \) has a 3-cycle \((u, v, w)\), then we can define a new tournament \( D'_{u,v,w} \) as follows: The vertices of \( D'_{u,v,w} \) shall be the same as those of \( D \), except that the three arcs \((u, v), (v, w)\) and \((u, w)\) are replaced by the three new arcs \((v, u), (w, v)\) and \((u, w)\). (Visually speaking, \( D'_{u,v,w} \) is obtained from \( D \) by turning the arrows on the arcs \((u, v), (v, w)\) and \((w, u)\) around.) We say that the new tournament \( D'_{u,v,w} \) is obtained from the old tournament \( D \) by a 3-cycle reversal operation.

Now, let \( V \) be a finite set, and let \( E \) and \( F \) be two tournaments with vertex set \( V \). Prove that \( F \) can be obtained from \( E \) by a sequence of 3-cycle reversal operations if and only if each \( v \in V \) satisfies \( \deg^-_E (v) = \deg^-_F (v) \). (Note that a sequence may be empty, which allows handling the case \( E = F \) even if \( E \) has no 3-cycles to reverse.)

A tournament \( D = (V, A) \) is called transitive if it has no 3-cycles.

Exercise 7. If a tournament \( D = (V, A) \) has three distinct vertices \( u, v \) and \( w \) satisfying \((u, v) \in A \) and \((v, w) \in A \), then we can define a new tournament \( D''_{u,v,w} \) as follows: The vertices of \( D''_{u,v,w} \) shall be the same as those of \( D \). The arcs of \( D''_{u,v,w} \) shall be the same as those of \( D \), except that the two arcs \((u, v)\) and \((v, w)\) are replaced by the two new arcs \((v, u)\) and \((w, v)\). We say that the new tournament \( D''_{u,v,w} \) is obtained from the old tournament \( D \) by a 2-path reversal operation.

Let \( D \) be any tournament. Prove that there is a sequence of 2-path reversal operations that transforms \( D \) into a transitive tournament.