1 Exercise 1

1.1 Problem

Let $G$ and $H$ be two simple graphs. The Cartesian product of $G$ and $H$ is a new simple graph, denoted $G \times H$, which is defined as follows:

- The vertex set $V(G \times H)$ of $G \times H$ is the Cartesian product $V(G) \times V(H)$.
- A vertex $(g, h)$ of $G \times H$ is adjacent to a vertex $(g', h')$ of $G \times H$ if and only if we have
  - either $g = g'$ and $hh' \in E(H)$,
  - or $h = h'$ and $gg' \in E(G)$.

(In particular, exactly one of the two equalities $g = g'$ and $h = h'$ has to hold when $(g, h)$ is adjacent to $(g', h')$.)

(a) Recall the $n$-dimensional cube graph $Q_n$ defined for each $n \in \mathbb{N}$. (Its vertices are $n$-tuples $(a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$, and two such vertices are adjacent if and only if
they differ in exactly one entry.) Prove that $Q_n \cong Q_{n-1} \times Q_1$ for each positive integer $n$. (Thus, $Q_n$ can be obtained from $Q_1$ by repeatedly forming Cartesian products; i.e., it is a “Cartesian power” of $Q_1$.)

(b) Assume that each of the graphs $G$ and $H$ has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian path.

(c) Assume that both numbers $|V(G)|$ and $|V(H)|$ are > 1, and that at least one of them is even. Assume again that each of the graphs $G$ and $H$ has a Hamiltonian path. Prove that $G \times H$ has a Hamiltonian cycle.

1.2 Solution to parts (a) and (b)

Proof of Part (a). To show that $Q_n \cong Q_{n-1} \times Q_1$, I will first show that their vertex sets are the same, provided that we identify each pair $((x_1, x_2, \ldots, x_{n-1}), x_n) \in \{0, 1\}^{n-1} \times \{0, 1\}$ with the $n$-tuple $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$. Then, we will show that their edge sets are the same.

The equality of their vertex sets follows directly from the definition of $V(Q_{n-1} \times Q_1)$:

\[
V(Q_{n-1} \times Q_1) = \{0, 1\}^{n-1} \times \{0, 1\} = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in \{0, 1\} \text{ for } 1 \leq i \leq n-1, \text{ and } x_n \in \{0, 1\}\} = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in \{0, 1\} \text{ for } 1 \leq i \leq n\} = \{0, 1\}^n = V(Q_n).
\]

To show the equality of the edge sets, I will first show that $E(Q_{n-1} \times Q_1) \subseteq E(Q_n)$, then that $E(Q_n) \subseteq E(Q_{n-1} \times Q_1)$. Let $\{(x_1, y_1), (x_2, y_2)\} \in E(Q_{n-1} \times Q_1)$, where $x_1, x_2 \in V(Q_{n-1})$ and $y_1, y_2 \in V(Q_1)$. Note that since $V(Q_{n-1} \times Q_1) = V(Q_n)$, $(x_1, y_1) \in V(Q_n)$ and $(x_2, y_2) \in V(Q_n)$. By the definition of $E(Q_{n-1} \times Q_1)$, there are two cases to consider:

- **Case 1:** $x_1 = x_2$ and $y_1, y_2 \in E(Q_1)$. (This is the “either” case from the definition.) In this case, $x_1 = x_2 = (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1})$. The graph $Q_1$ has only one edge: $\{0, 1\}$, so $\{(y_1, y_2) \mid y_1 \neq y_2\}$. Then $(x_1, y_1) = (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, y_1)$ and $(x_2, y_2) = (x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}, y_2)$ differ in only one entry ($y_1$ and $y_2$). Therefore, $\{(x_1, y_1), (x_2, y_2)\} \in E(Q_n)$.

- **Case 2:** $y_1 = y_2$ and $x_1, x_2 \in E(Q_{n-1})$. (This is the “or” case from the definition.) Since $x_1, x_2 \in E(Q_{n-1})$, the first $n-1$ entries in $(x_1, y_1)$ and $(x_2, y_2)$ differ in exactly one entry. But $y_1 = y_2$, so $(x_1, y_1)$ and $(x_2, y_2)$ differ in exactly one entry as well. Therefore, $\{(x_1, y_1), (x_2, y_2)\} \in E(Q_n)$.

Case 1 and Case 2 together show that $E(Q_{n-1} \times Q_1) \subseteq E(Q_n)$. It must now be shown that $E(Q_n) \subseteq E(Q_{n-1} \times Q_1)$. Let $\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n})\} \in E(Q_n)$. Since $(x_{1,1}, x_{1,2}, \ldots, x_{1,n})$ and $(x_{2,1}, x_{2,2}, \ldots, x_{2,n})$ must differ in exactly one entry, there are two cases to consider.
Case 1: \((x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}) = (x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})\) and \(x_{1,n} \neq x_{2,n}\). This implies that \(\{x_{1,n}, x_{2,n}\} = \{0, 1\} \in E(Q_1)\). \((x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}) = (x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})\) and \(\{x_{1,n}, x_{2,n}\} \in E(Q_1)\) satisfy the “either” condition for
\[
\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n})\} \in E(Q_{n-1} \times Q_1).
\]

Case 2: \(x_{1,n} = x_{2,n}\) and \((x_{1,1}, x_{1,2}, \ldots, x_{1,n-1})\) differs from \((x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})\) in exactly one entry. This implies that \(\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n-1}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n-1})\} \in E(Q_{n-1})\). Together with \(x_{1,n} = x_{2,n}\), this satisfies the “or” condition for
\[
\{(x_{1,1}, x_{1,2}, \ldots, x_{1,n}), (x_{2,1}, x_{2,2}, \ldots, x_{2,n})\} \in E(Q_{n-1} \times Q_1).
\]

Case 1 and Case 2 together show that \(E(Q_n) \subseteq E(Q_{n-1} \times Q_1)\). Therefore, \(E(Q_n) = E(Q_{n-1} \times Q_1)\). Since \(V(Q_{n-1} \times Q_1) = V(Q_n)\) and \(E(Q_n) = E(Q_{n-1} \times Q_1)\), it follows that \(Q_n \cong Q_{n-1} \times Q_1\). □

Proof of Part (b). Let \(n = |V(G)|\) and \(m = |V(H)|\). Since both \(G\) and \(H\) have a Hamiltonian path, there is a listing \((v_1, v_2, \ldots, v_n)\) of the vertices of \(G\) such that \(v_i v_{i+1} \in E(G)\) for all \(1 \leq i \leq n - 1\), and a listing \((w_1, w_2, \ldots, w_m)\) of the vertices of \(H\) such that \(w_i w_{i+1} \in E(H)\) for all \(1 \leq i \leq m - 1\). It follows from the definition of \(E(G \times H)\) that the following holds:

- \(\{(v_i, w_j), (v_i, w_{j+1})\} \in E(G \times H)\) for all \(1 \leq i \leq n\) and \(1 \leq j \leq m - 1\), and
- \(\{(v_i, w_j), (v_{i+1}, w_j)\} \in E(G \times H)\) for all \(1 \leq i \leq n - 1\) and \(1 \leq j \leq m\).

Thus we may construct a Hamiltonian path as below:
\[
((v_1, w_1), (v_1, w_2), \ldots, (v_1, w_m), (v_2, w_m), (v_2, w_{m-1}), \ldots, (v_2, w_1), (v_3, w_1), (v_3, w_2),
\ldots, (v_{n-1}, w_a), (v_n, w_a), \ldots, (v_n, w_b)),
\]

where \(a = 1\) if \(n \equiv 1\) mod 2 and \(a = m\) if \(n \equiv 0\) mod 2; and \(b = m\) if \(n \equiv 1\) mod 2 and \(b = 1\) if \(n \equiv 0\) mod 2. It is easily verified that each consecutive pair of vertices is adjacent in \(G \times H\), and that each of the \(n \cdot m\) vertices of \(G \times H\) appears exactly once. Indeed, this path fully traverses the \(n \times m\) matrix \(M\) where the entry \(m_{i,j} = (v_i, w_j)\), descending row by row in alternating (left/right) directions. □

2 Exercise 2

2.1 Problem

Let \(n\) be a positive integer. Recall that \(K_n\) denotes the complete graph on \(n\) vertices. This is the graph with vertex set \(V = \{1, 2, \ldots, n\}\) and edge set \(P_2(V)\) (so that each two distinct vertices are connected). Find Eulerian circuits for the graphs \(K_3, K_5,\) and \(K_7\).
2.2 Solution

We shall represent walks as lists of edges, omitting the vertices.

- For $K_3$, the edge set contains only the three edges $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$. The edges in this order are already an Eulerian circuit: $\{(1, 2), (2, 3), (3, 1)\}$.

- In the representation of the graph $K_5$ below, an Eulerian circuit can be created by starting at 1 and first following the edges clockwise around the outer pentagon back to 1, then following the edges of the inner pentagram back to 1:

$$\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3), (3, 5), (5, 2), (2, 4), (4, 1)\}.$$

Note that all $\binom{5}{2} = 10$ edges are included, with the first 5 edges $\{i, j\}$ having $(j - i) \equiv 1 \mod 5$ and the next 5 edges having $(j - i) \equiv 2 \mod 5$.

- In the representation of the graph $K_7$ below, an Eulerian circuit can be created by starting at 1 and first following the edges clockwise around the outer heptagon back to 1, then following the edges of the first inner heptagram clockwise by skipping 1 vertex each time back to 1, and finally following the other inner heptagram clockwise by skipping two vertices each time back to 1:

$$\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 1), (1, 3), (3, 5), (5, 7), (7, 2), (2, 4), (4, 6), (6, 1), (1, 4), (4, 7), (7, 3), (3, 6), (6, 2), (2, 5), (5, 1)\}.$$

Note that all $\binom{7}{2} = 21$ edges are included, with the first 7 edges $\{i, j\}$ having $(j - i) \equiv 1 \mod 7$, the next 7 edges having $(j - i) \equiv 2 \mod 7$, and the final 7 edges having $(j - i) \equiv 3 \mod 7$. 
4 EXERCISE 4

4.1 PROBLEM

Let $D$ be a digraph. Show that $\sum_{v \in V(D)} \deg^-(v) = \sum_{v \in V(D)} \deg^+(v)$.

4.2 SOLUTION

Proof. Let $V = V(D)$ and $A = A(D)$. (We use the notation $A(D)$ for the set of all arcs of $D$.)

Using the definition of $\deg^-(v)$, the sum on the left hand side can be written as

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} |\{a \in A \mid v \text{ is the target of } a\}|.$$

Next, using Proposition 0.3.a from homework set 1, the sum can be rewritten again as

$$\sum_{v \in V} |\{a \in A \mid v \text{ is the target of } a\}| = \sum_{v \in V} \sum_{u \in V} [(u, v) \in A].$$

Finally, the order of the summation can be flipped, and we can reverse the above process to arrive at the conclusion:

$$\sum_{u \in V} \sum_{v \in V} [(u, v) \in A] = \sum_{u \in V} \sum_{v \in V} [(u, v) \in A]$$

$$= \sum_{u \in V} |\{a \in A \mid u \text{ is the source of } a\}|$$

$$= \sum_{u \in V} \deg^+(u) = \sum_{v \in V} \deg^+(v).$$

\[\square\]

5 EXERCISE 5

5.1 PROBLEM

Let $D = (V, A)$ be a tournament. Set $n = |V|$ and $m = \sum_{v \in V} \binom{\deg^+(v)}{2}$. 

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(a) Show that \( m = \sum_{v \in V} \binom{\text{deg}^-(v)}{2} \).

(b) Show that the number of 3-cycles in \( D \) is \( 3 \cdot \left( \binom{n}{3} - m \right) \).

5.2 Solution

Proof of Part (a). To begin, note that for every \( v \in V \), by the definition of a tournament we have that for each of the \( n - 1 \) vertices \( u \in V \) such that \( u \neq v \), either \((u, v) \in A \) or \((v, u) \in A \) but not both. Thus we have for each \( v \in V \) the equality \( \text{deg}^+(v) + \text{deg}^-(v) = n - 1 \), and therefore

\[
\text{deg}^+(v) = n - 1 - \text{deg}^-(v).
\]

(1)

Also, since there is exactly one arc between each pair of vertices, we have \( |A| = \sum_{u \in V} \sum_{v \in V} [(u, v) \in A] = \sum_{v \in V} \text{deg}^+(v) \), so we have

\[
\sum_{v \in V} \text{deg}^+(v) = \binom{n}{2} = \frac{n \cdot (n - 1)}{2}.
\]

(2)

Using (1), we can rewrite the definition of \( m \) as

\[
m = \sum_{v \in V} \binom{n - 1 - \text{deg}^-(v)}{2}.
\]

Expanding the binomial coefficient and performing algebra:

\[
m = \sum_{v \in V} \binom{n - 1 - \text{deg}^-(v)}{2}(n - 2 - \text{deg}^-(v))
\]

\[
= \frac{1}{2} \sum_{v \in V} (\text{deg}^-(v))^2 - \text{deg}^-(v) + \frac{1}{2} \sum_{v \in V} (4 \cdot \text{deg}^-(v) - 2n \cdot \text{deg}^-(v) + n^2 - 3n + 2)
\]

\[
= \sum_{v \in V} \frac{\text{deg}^-(v) (\text{deg}^-(v) - 1)}{2} + (2 - n) \sum_{v \in V} \text{deg}^-(v) + \frac{1}{2} \sum_{v \in V} (n^2 - 3n + 2).
\]

We can now make use of (2) and the fact that \( |V| = n \) to evaluate the second and third sums:

\[
= \sum_{v \in V} \binom{\text{deg}^-(v)}{2} + \frac{(2 - n)n(n - 1)}{2} + \frac{n(n - 1)(n - 2)}{2}
\]

\[
= \sum_{v \in V} \binom{\text{deg}^-(v)}{2}.
\]
Proof of Part (b). Let

\[ S = \{ s \in P_3(V) \mid \text{the sub-digraph } D \mid_s \text{ contains at least one 3-cycle} \} \quad \text{and} \]
\[ T = \{ t \in P_3(V) \mid \text{the sub-digraph } D \mid_t \text{ contains no 3-cycles} \}. \]

(Here, \( D \mid_s \) denotes the sub-digraph of \( D \) obtained by removing all vertices that don’t lie in \( s \), and removing all arcs that don’t connect two vertices in \( s \).)

Consider an arbitrary \( s \in S \). There exists a 3-cycle \((s_1, s_2, s_3)\) in the sub-digraph \( D \mid_s \), so \((s_1, s_2), (s_2, s_3), \) and \((s_3, s_1)\) all belong to \( A \). But then \((s_2, s_3, s_1)\) and \((s_3, s_1, s_2)\) are also 3-cycles in \( D \). On the other hand, since \( D \) is a tournament, none of \((s_2, s_1), (s_3, s_2), \) or \((s_1, s_3)\) belong to \( A \). Hence none of \((s_3, s_2, s_1), (s_2, s_1, s_3), \) or \((s_1, s_3, s_2)\) are 3-cycles in \( D \). Therefore, for each \( s \in S \), the sub-digraph \( D \mid_s \) contains exactly three 3-cycles. Hence, the total number of 3-cycles in \( D \) is \( 3|S| \) (because clearly, each 3-cycle in \( D \) must be contained in \( D \mid_s \) for a unique \( s \in S \)). It thus remains to be shown that \( |S| = \binom{n}{3} - m \).

The problem can be simplified further by noting that \(|P_3(V)| = \binom{n}{3}\). Since \(|S| = |P_3(V)| - |T|\), the goal is reduced to showing that \(|T| = m\).

Consider an arbitrary \( t = \{t_1, t_2, t_3\} \in T \). Since the sub-digraph \( D \mid_t \) must contain exactly one arc between each pair of vertices, we have \( \sum_{i=1}^{3} \deg_{D \mid_t}(t_i) = 3 \).

Now, we can’t have \( \deg_{D \mid_t}(t_i) = 1 \) for all \( 1 \leq i \leq 3 \), or we would have a 3-cycle. Nor can we have \( \deg_{D \mid_t}(t_i) = 3 \) for any \( 1 \leq i \leq 3 \), since this would contradict the definition of a tournament. Therefore we must have \( \deg_{D \mid_t}(t_i) = 2 \) for exactly one \( 1 \leq i \leq 3 \). There are arcs from the other two elements of \( t \) having \( t_i \) as their target.

Hence, each subset \( t \in T \) has exactly one element \( t_i \) satisfying \( \deg_{D \mid_t}(t_i) = 2 \), and there are arcs from the other two elements of \( t \) having \( t_i \) as their target. Consequently, each \( t \in T \) gives rise to two arcs of \( D \) having a common target. Conversely, any two arcs of \( D \) having a common target are obtained from exactly one \( t \in T \). To see this, pick any two arcs \((u, v) \in A \) and \((w, v) \in A \) with a common target \( v \). The vertices of these arcs define a unique element of \( P_3(V) \), namely \( \{u, v, w\} \). Clearly the sub-digraph \( D \mid_{\{u,v,w\}} \) does not contain a 3-cycle, so \( \{u, v, w\} \in T \).

Each choice of two arcs having a common target defines exactly one element of \( T \), and each element of \( T \) contains exactly one such vertex. Hence, the elements in \( T \) can be counted by totaling the distinct (unordered) pairs of arcs in \( A \) with a common vertex as their target. But this is precisely what \( m = \sum_{v \in V} \binom{\deg^{-}(v)}{2} \) counts (because we can count them by first choosing their common target \( v \), and then we have \( \binom{\deg(v)}{2} \) options for choosing the two arcs). So we have \(|T| = m\), and therefore

\[ \# \text{ 3-cycles in } D = 3(|P_3(V)| - |T|) = 3 \left( \binom{n}{3} - m \right). \]
7 Exercise 7

7.1 Problem
Let $D$ be any tournament. Prove that there is a sequence of 2-path reversal operations that transforms $D$ into a transitive tournament.

7.2 Solution

Proof. Let $V = V(D)$, $A = A(D)$, and $n = |V|$. If $D$ contains no 3-cycles, it is already transitive.

It suffices to show that as long as a 3-cycle exists, a 2-path reversal operation can be chosen that will decrease the number of 3-cycles. Assume that $D$ contains at least one 3-cycle. By the results of Exercise 5 (b), the number of 3-cycles is given by

$$3 \left( \binom{n}{3} - \sum_{v \in V} \binom{\deg^+(v)}{2} \right) = 3 \left( \binom{n}{3} - \sum_{v \in V} \binom{\deg^{-}(v)}{2} \right)$$

(since Exercise 5 (a) yields $\sum_{v \in V} \binom{\deg(v)}{2} = \sum_{v \in V} \binom{\deg^{-}(v)}{2}$). Hence, in order to decrease the number of 3-cycles, we need to increase the sum $\sum_{v \in V} \binom{\deg^{-}(v)}{2}$.

Pick any 3-cycle $(u, v, w)$ in $D$. Consider two cases: either the indegrees of $u$, $v$, and $w$ are all equal, or they are not all equal.

- **Case 1**: $\deg^-(u) = \deg^-(v) = \deg^-(w)$. Set $d = \deg^-(u)$. In this case, performing the 2-path reversal that replaces the arcs $(u, v)$ and $(v, w)$ with $(v, u)$ and $(w, v)$ will increase $\deg^-(u)$ by 1, decrease $\deg^-(w)$ by 1, and leave the indegrees of all other vertices unchanged.

Let $m$ be the value of the sum $\sum_{v \in V} \binom{\deg^-(v)}{2}$ before the 2-path reversal, and $m''$ be the value after. Then we have:

$$m'' = m + \left[ \binom{d+1}{2} - \binom{d}{2} \right] + \left[ \binom{d-1}{2} - \binom{d}{2} \right] = m + (d) + (1 - d) = m + 1.$$ 

Thus this 2-path reversal operation reduces the number of 3-cycles by 3 (since the number of 3-cycles is $3 \left( \binom{n}{3} - \sum_{v \in V} \binom{\deg^-(v)}{2} \right)$).
Case 2: \(\deg^-(u), \deg^-(v),\) and \(\deg^-(w)\) are not all equal. There are three 3-cycles containing the vertices \(u, v,\) and \(w.\) Since the indegrees are not all equal, at least one 3-cycle \((x, y, z)\) among the three has \(\deg^-(x) > \deg^-(z).\) Fix such a 3-cycle, and set \(d = \deg^-(x)\) and \(c = \deg^-(z).\) Performing the 2-path reversal operation that replaces the arcs \((x, y)\) and \((y, z)\) with \((y, x)\) and \((z, y)\) will increase \(\deg^-(x)\) by 1, decrease \(\deg^-(z)\) by 1, and leave the indegrees of all other vertices unchanged. As in Case 1, let \(m\) be the value of the sum \(\sum_{v \in V} \binom{\deg^-(v)}{2}\) before the 2-path reversal, and \(m''\) be the value after. We now have

\[m'' = m + \left(\binom{d+1}{2} - \binom{d}{2}\right) + \left(\binom{c-1}{2} - \binom{c}{2}\right) = m + (d) + (1 - c).\]

Since \(d > c,\) this yields \(m'' - m \geq 2.\) Hence this 2-path reversal reduces the number of 3-cycles by at least 6 (since the number of 3-cycles is \(3\left(\binom{n}{3} - \sum_{v \in V} \binom{\deg^-(v)}{2}\right)\)).

In either case, a 2-path reversal operation on the vertices of the 3-cycle can be chosen to reduce the total number of 3-cycles. Since there are a finite number of 3-cycles, a finite number of 2-path reversal operations will reduce the number of 3-cycles to 0. \qed