0.1. Exercise \[1\] the distances between four points in a graph

**Exercise 1.** Let \( G \) be a connected multigraph. Let \( x, y, z \) and \( w \) be four vertices of \( G \).

Assume that the two largest ones among the three numbers \( d(x, y) + d(z, w) \), \( d(x, z) + d(y, w) \) and \( d(x, w) + d(y, z) \) are **not** equal.

Prove that \( G \) has a cycle of length \( \leq d(x, z) + d(y, w) + d(x, w) + d(y, z) \).

**[Hint:** This is a strengthening of Exercise 6 on midterm #2. Try deriving it by applying the latter exercise to a strategically chosen submultigraph of \( G \).]

0.2. Exercise \[2\] more examples of chromatic polynomials

See Exercise 4 on midterm #2 for the definition and the properties of the chromatic polynomial.
Exercise 2. (a) Let \( n > 1 \) be an integer. Prove that the chromatic polynomial of the cycle graph \( C_n \) is

\[
\chi_{C_n} = (x - 1)^n + (-1)^n (x - 1).
\]

[Hint: Induct over \( n \).]

(b) Let \( g \in \mathbb{N} \). Let \( G \) be the simple graph whose vertices are the \( 2g + 1 \) integers

\(-g, -g + 1, \ldots, g - 1, g,\)

and whose edges are

\[
\begin{align*}
\{0, i\} & \quad \text{for all } i \in \{1, 2, \ldots, g\}; \\
\{0, -i\} & \quad \text{for all } i \in \{1, 2, \ldots, g\}; \\
i, -i & \quad \text{for all } i \in \{1, 2, \ldots, g\}
\end{align*}
\]

(these are \( 3g \) edges in total).

Compute the chromatic polynomial \( \chi_G \) of \( G \).

[Here is how \( G \) looks like in the case when \( g = 4 \):]

0.3. Exercise 3: unhappy marriages in a lopsided bipartite graph

Exercise 3. Let \((G; X, Y)\) be a bipartite graph such that \(|Y| \geq 2|X| - 1\). Prove that there exists an injective map \( f : X \to Y \) such that each \( x \in X \) satisfies one of the following two statements:

- **Statement 1**: The vertices \( x \) and \( f(x) \) of \( G \) are adjacent.
- **Statement 2**: There exists no \( x' \in X \) such that the vertices \( x \) and \( f(x') \) of \( G \) are adjacent.
(The intuition behind Exercise 3 is something along the lines of: If there are significantly more ladies than there are gentlemen, then we can marry all gentlemen off in such a way that each gentleman is either married to a lady he likes, or none of the ladies he likes is married.)

0.4. Exercise 4: crossing numbers of \( n \)-matchings

The purpose of the next exercise is to fill in some details (namely, the verification that the signs match) in our proof of one of the basic properties of the Pfaffian (the fact that the square of the Pfaffian is the determinant).

For this section, fix an \( n \in \mathbb{N} \). We shall use the following notations:

- We use the Iverson bracket notation (i.e., the truth value of a statement \( A \) will be called \([A]\)).
- We set \([k] = \{1, 2, \ldots, k\}\) for each \( k \in \mathbb{N} \). (This does not conflict with the Iverson bracket notation, since \( k \) is not a statement.)
- We let \( S_n \) denote the set of all the \( n! \) permutations of the set \([n]\). (These permutations are the bijective maps \([n] \to [n]\).)
- The 2-element subsets of \([n]\) will be called edges. (This makes sense, since they are the edges of the complete graph \( K_n \). We shall not consider any other graphs in this section.)
- The length \( \ell(\sigma) \) of a permutation \( \sigma \in S_n \) is the number of all pairs \( (i, j) \in [n]^2 \) satisfying \( i < j \) and \( \sigma(i) > \sigma(j) \). For example, the permutation in \( S_4 \) that sends 1, 2, 3, 4 to 3, 1, 4, 2 (respectively) has length 3.
- The sign of a permutation \( \sigma \in S_n \) is the integer \((-1)^{\ell(\sigma)}\). It is denoted by \((-1)^{\sigma}\) (or by \(\text{sign } \sigma\)).
- A matching shall mean a set \( M \subseteq \mathcal{P}_2([n]) \) (that is, a set \( M \) of edges) with the property that each element of \([n]\) belongs to at most one element of \( M \) (equivalently, with the property that the edges in \( M \) are disjoint). In other words, a matching is a matching of the complete graph \( K_n \). For instance, if \( n \geq 7 \), then \( \{\{1, 4\}, \{2, 7\}\} \) is a matching (but \( \{\{1, 4\}, \{4, 7\}\} \) is not).
- A perfect matching shall mean a matching \( M \) such that each element of \([n]\) belongs to exactly one element of \( M \). In other words, a perfect matching is a perfect matching of the complete graph \( K_n \). Note that such a perfect matching exists only if \( n \) is even; in this case, there are \((n - 1)(n - 3)(n - 5) \cdots 1\) many such perfect matchings, and each of them has size \( n/2 \).
• If \( e \) and \( f \) are two disjoint edges, then we say that \( e \) crosses \( f \) if and only if exactly one element of \( e \) lies between the two elements of \( f \). (For example, \( \{2, 6\} \) crosses \( \{3, 7\} \), but does not cross \( \{3, 4\} \) or \( \{1, 7\} \) or \( \{7, 8\} \).)

It is easy to check that the relation “crosses” is symmetric: i.e., if \( e \) and \( f \) are two disjoint edges, then \( e \) crosses \( f \) if and only if \( f \) crosses \( e \).

The geometrical meaning of “crossing” is simple: If we draw a regular polygon with \( n \) vertices \( v_1, v_2, \ldots, v_n \) in the plane, then an edge \( \{i, j\} \) crosses an edge \( \{k, \ell\} \) if and only if the diagonal \( v_i v_j \) crosses the diagonal \( v_k v_\ell \).

Also easy to verify is the following simple fact: If \( \{i, j\} \) and \( \{k, \ell\} \) are two disjoint edges, then

\[
\{i, j\} \text{ crosses } \{k, \ell\} \equiv [i > k] + [i > \ell] + [j > k] + [j > \ell] \mod 2. \tag{1}
\]

(You can use these facts without proof.)

• If \( M \) is a matching, then the crossing number \( \text{xing} M \) of \( M \) is defined to be the number of all unordered pairs \( \{e, f\} \) of two disjoint edges \( e \in M \) and \( f \in M \) such that \( e \) crosses \( f \). (Notice that we are counting unordered pairs; hence, we do not count \( \{e, f\} \) and \( \{f, e\} \) twice.)

For example, the crossing number of the matching \( \{\{1, 5\}, \{2, 6\}, \{3, 4\}\} \) is 1, since the only two edges crossing in this matching are \( \{1, 5\} \) and \( \{2, 6\} \).

**Exercise 4.** Let \( n \in \mathbb{N} \) be even. Let \( \sigma \in S_n \) be a permutation.

(a) Show that the perfect matching

\[
M_\sigma = \{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \{\sigma(5), \sigma(6)\}, \ldots, \{\sigma(n-1), \sigma(n)\}\}
\]

satisfies

\[
(-1)^{\text{xing}(M_\sigma)} (-1)^k = (-1)^\sigma,
\]

where \( k \) is the number of \( i \in \{1, 2, \ldots, n/2\} \) satisfying \( \sigma(2i-1) > \sigma(2i) \).

(b) Let

\[
M = \{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_{n/2}, b_{n/2}\}\}
\]

be any perfect matching. Define a new perfect matching \( \sigma M \) by

\[
\sigma M = \{\{\sigma(a_1), \sigma(b_1)\}, \{\sigma(a_2), \sigma(b_2)\}, \ldots, \{\sigma(a_{n/2}), \sigma(b_{n/2})\}\}.
\]

Let \( p \) be the number of \( i \in \{1, 2, \ldots, n/2\} \) satisfying \( a_i > b_i \).

Let \( q \) be the number of \( i \in \{1, 2, \ldots, n/2\} \) satisfying \( \sigma(a_i) > \sigma(b_i) \).

Prove that

\[
(-1)^{\text{xing}(\sigma M)} (-1)^p = (-1)^\sigma (-1)^{\text{xing} M} (-1)^q.
\]

---

\(^2\)The word “between” is to be understood in the sense of inequalities: An integer \( p \) is said to lie between two integers \( u \) and \( v \) if \( \min\{u, v\} < p < \max\{u, v\} \).
0.5. Exercise 5: pushing sources in orientations 1

See homework set #5 for the concepts of orientations and of acyclic orientations.

**Definition 0.1.** Let $G = (V, E, \psi)$ be a multigraph.

Let $\phi$ be an orientation of $G$.

A vertex $v \in V$ is said to be a **source** of $\phi$ if no arc of the multidigraph $(V, E, \phi)$ has target $v$. Exercise 6 (a) on homework set #5 shows that if $\phi$ is acyclic and if $V \neq \emptyset$, then there exists a source of $\phi$.

If $v$ is a source of $\phi$, then we can define a new orientation $\phi'$ of $G$ as follows:

- For each $e \in E$ satisfying $v \in \psi(e)$, we set $\phi'(e) = (u, v)$, where $u$ is chosen such that $\phi(e) = (v, u)$.

- For all other $e \in E$, we set $\phi'(e) = \phi(e)$.

(Roughly speaking, this simply means that $\phi'$ is obtained by $\phi$ by reversing the directions of all edges that contain $v$.) We say that this new orientation $\phi'$ is obtained from $\phi$ by **pushing the source $v$**.
Example 0.2. Let $G = (V, E, \psi)$ be the following multigraph:

Consider the orientation $\phi$ of $G$ for which the multidigraph $(V, E, \phi)$ looks as follows:

(Formally speaking, this is the orientation $\phi$ that sends the edges $a, b, c, d, e, f$ to the pairs $(1, 2), (3, 2), (1, 4), (3, 5), (4, 5), (5, 4)$, respectively.)

This orientation $\phi$ has two sources 1 and 3. We can transform this orientation by pushing the source 1; this results in the following orientation $\phi'$ (shown here by drawing the multidigraph $(V, E, \phi')$):

This new orientation $\phi'$ has a single source, 3. If we push this source, we obtain a new orientation $\phi''$, which looks as follows (again, represented by the multidigraph $(V, E, \phi'')$):

This orientation $\phi''$, in turn, has a single source, 2. If we push this source, we
obtain a new orientation $\phi'''$, which looks as follows (again, represented by the multidigraph $(V, E, \phi''')$):

\[
\begin{array}{c}
\rightarrow \quad \uparrow \\
\downarrow \quad \rightarrow \\
\uparrow \quad \downarrow \\
\rightarrow \quad \uparrow \\
\end{array}
\]

This orientation $\phi'''$ has no sources, and thus cannot be transformed any further by pushing sources.

The preceding example might have suggested some questions: For example, given an orientation of a multigraph, can we keep pushing sources indefinitely, or will we eventually end up at an orientation that has no more sources? The following is easy to see:

**Proposition 0.3.** Let $\phi$ be an acyclic orientation of a multigraph $G = (V, E, \psi)$. Let $v$ be a source of $\phi$. Then, the orientation obtained from $\phi$ by pushing the source $v$ is again acyclic.

This proposition shows that if we start with an acyclic orientation of a multigraph (with at least one vertex), then we can keep pushing sources indefinitely (since the orientation always remains acyclic, and thus there always will be sources to push). The next exercise (specifically, Exercise 5 (b)) yields a converse (for connected multigraphs): If we can keep pushing sources indefinitely, then our orientation must have been acyclic.

**Exercise 5.** Let $G = (V, E, \psi)$ be a connected multigraph. Set $n = |V|$ and $h = |E|$.

Let $(\phi_0, \phi_1, \ldots, \phi_k)$ be a sequence of orientations of $G$, and let $(v_1, v_2, \ldots, v_k)$ be a sequence of vertices of $G$ such that for each $i \in \{1, 2, \ldots, k\}$, the orientation $\phi_i$ is obtained from $\phi_{i-1}$ by pushing the source $v_i$ (in particular, this is saying that $v_i$ is a source of $\phi_{i-1}$).

Assume that $k \geq \binom{n + h - 1}{n - 1}$.

(a) Prove that each vertex of $G$ appears at least once in the sequence $(v_1, v_2, \ldots, v_k)$.

(b) Prove that the orientations $\phi_0, \phi_1, \ldots, \phi_k$ are acyclic.

(At this point, let me remind you that you can freely use the exercises on the previous homework sets.)
0.6. Exercise 6: pushing sources in orientations 2

Exercise 6. Let $G = (V, E, \psi)$ be a tree. Let $\alpha$ and $\beta$ be two orientations of $G$. Prove that $\beta$ can be obtained from $\alpha$ by repeatedly pushing sources.

(More rigorously: Prove that there exists a sequence $(\phi_0, \phi_1, \ldots, \phi_k)$ of orientations of $G$, and a sequence $(v_1, v_2, \ldots, v_k)$ of vertices of $G$ such that

- we have $\phi_0 = \alpha$ and $\phi_k = \beta$, and
- for each $i \in \{1, 2, \ldots, k\}$, the orientation $\phi_i$ is obtained from $\phi_{i-1}$ by pushing the source $v_i$ (in particular, this is saying that $v_i$ is a source of $\phi_{i-1}$).

Notice that $k$ is allowed to be 0.)