Def. Let $G$ be a connected graph.

(see [Gelman].)

\[
\frac{i(T - 1)}{n - 2} \leq \frac{i_1(T - 1)}{n_1 - 2} + \cdots + \frac{i_n(T - 1)}{n_n - 2}
\]

with vertex set $I = \{1, 2, \ldots, n\}$. Then, the # of these
with $d_1 + d_2 + \cdots + d_n = 2(n - 1)$.

If $G_{i_1, d_1}, G_{i_2, d_2}, \ldots, G_{i_n, d_n}$ be $n$ nonempty
    \[ (n - 2)! \]

edges from $G$ such that $G$ is a tree (informally).

A spanning tree of $G$ is a way to remove

but also

has spanning tree

Example:

(see [Gelman].)
Theorem 1.2: \( \Delta \) graphs have \( \Delta \) spanning trees.

Theorem 1.3: Any connected graph \( G \) has a spanning tree.

\( K_n = ([n], \theta ([n])) \).

For each \( n \geq 2 \), consider the complete graph \( K_n \).

Then, the \# of spanning trees of \( G \) is \( \Delta \).

\[ \frac{n-1}{2} \]

Let \( G \) be a connected graph with \( n \) vertices. If \( G \) is complete, then \( G \) is a spanning tree.

\[ n \in \mathbb{Z}^+ \]

Example: For each \( n \geq 2 \), consider the complete graph \( K_n \).

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Then, the \# of spanning trees of \( G \) is \( \Delta \).

\[ \frac{n-1}{2} \]
Elementary linear algebra says that \( \det L = u - 2 \),

\[
\begin{pmatrix}
1 & u - 2 & \\
1 & 1 & u - 2 \\
1 & 1 & u - 2 \\
\end{pmatrix}
\]

where \( \det L = 7 \).
If \( G \) has no Eulerian circuit, then either of the vertices with odd degrees.

A circuit is an Eulerian graph.

Given a graph \( G = (V, E) \), an Eulerian circuit of \( G \) exists if and only if every vertex in \( G \) has an even degree.
(c) If $G = \begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (1,1) {c};
\node (d) at (0,1) {d};
\node (e) at (0.5,0.5) {e};
\node (f) at (1.5,0.5) {f};
\node (g) at (2,1) {g};
\node (h) at (1.5,1) {h};
\draw (a) -- (d);
\draw (b) -- (c);
\draw (a) -- (e);
\draw (b) -- (f);
\draw (c) -- (e);
\draw (d) -- (f);
\draw (a) -- (g);
\draw (b) -- (c);
\draw (c) -- (g);
\draw (d) -- (f);
\draw (e) -- (h);
\draw (f) -- (h);
\draw (g) -- (h);
\end{tikzpicture}$

then $G$ has no Eulerian circuit.

But $\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (1,1) {c};
\node (d) at (0,1) {d};
\node (e) at (0.5,0.5) {e};
\node (f) at (1.5,0.5) {f};
\node (g) at (2,1) {g};
\node (h) at (1.5,1) {h};
\draw (a) -- (d);
\draw (b) -- (c);
\draw (a) -- (e);
\draw (b) -- (f);
\draw (c) -- (e);
\draw (d) -- (f);
\draw (a) -- (g);
\draw (b) -- (c);
\draw (c) -- (g);
\draw (d) -- (f);
\draw (e) -- (h);
\draw (f) -- (h);
\draw (g) -- (h);
\end{tikzpicture}$

does.

Thm. 13 (Euler & Hierholzer).

Let $G$ be a connected Eulerian circuit if and only if each vertex of $G$ has even degree.

Ex: $\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (1,1) {c};
\node (d) at (0,1) {d};
\node (e) at (0.5,0.5) {e};
\node (f) at (1.5,0.5) {f};
\node (g) at (2,1) {g};
\node (h) at (1.5,1) {h};
\draw (a) -- (d);
\draw (b) -- (c);
\draw (a) -- (e);
\draw (b) -- (f);
\draw (c) -- (e);
\draw (d) -- (f);
\draw (a) -- (g);
\draw (b) -- (c);
\draw (c) -- (g);
\draw (d) -- (f);
\draw (e) -- (h);
\draw (f) -- (h);
\draw (g) -- (h);
\end{tikzpicture}$

← this has no Euler circuit.

$\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (1,1) {c};
\node (d) at (0,1) {d};
\node (e) at (0.5,0.5) {e};
\node (f) at (1.5,0.5) {f};
\node (g) at (2,1) {g};
\node (h) at (1.5,1) {h};
\draw (a) -- (d);
\draw (b) -- (c);
\draw (a) -- (e);
\draw (b) -- (f);
\draw (c) -- (e);
\draw (d) -- (f);
\draw (a) -- (g);
\draw (b) -- (c);
\draw (c) -- (g);
\draw (d) -- (f);
\draw (e) -- (h);
\draw (f) -- (h);
\draw (g) -- (h);
\end{tikzpicture}$

← this one has.
7.7. **BIPARTITE GRAPHS & HALL'S THM.**

**Def.** A bipartite graph is a triple \((G; X, Y)\), where \(G = (V, E)\) is a simple graph, \(X\) and \(Y\) are two subsets of \(V\) such that:

- \(X \cup Y = V\), \(X \cap Y = \emptyset\);
- each edge \(e \in E\) has exactly one endpoint in \(X\) & exactly one endpoint in \(Y\).

**Ex:**

\[\begin{array}{c}
\text{X} \\
\text{Y}
\end{array}\]
A matching in a graph $G$ is a set of disjoint edges of $G$.

Let $S$ be a set of vertices of a graph $G$, let $M$ be a matching in $G$, and say that $M$ is $S$-complete if each vertex in $S$ is contained in a matching in $G$.

Examples:

If $G = \{1, 2, 3, 4, 5\}$, then $\{1\}$ is a matching, but $\{\{2, 3\}\}$ is not.

If $G = \{1, 2, 3, 4, 5\}$, then $\{2, 3, 3\}$ is a matching, $\{1, 4, 2\}$ is not.

$\{1, 4, 2, 3, 4, 3, 3\}$ is a matching.
For an elementary proof, see [Lecture, Thm 4.2].

where $V = \{v_1, v_2, \ldots, v_n\}$.

Then, $|N(v)| \geq \frac{n}{2}$, where $N(v) = \{u \in V \mid v \neq u\}$ has a

neighbor in $V$. Let $(G, X, Y)$ be a bipartite graph. Then, $X$-complete matching $M$ is

if and only if each subset $U \subseteq X$ satisfies

Thm. 14 ("Hall's marriage theorem").

Let $(G, X, Y)$ be a

No, because the 3 vertices $2, 3, 4$ have only 2 edges.

Does $G$ have a $X$-complete matching?

\[ G = \]

Ex.
\[ u \rightarrow v \] is the target of \( u \). Let \( \text{hw} \) as \( u \rightarrow v \) if \( u \in A \) with \( \text{hw}(u) = (v, a) \), then \( u \) is the source of \( v \), and if \( \text{hw}(u) \) is the set of \( v \), an edge \( u \rightarrow v \) is the edge of \( A \).

The vertices of the multigraph \((V, A, p)\) are the sets of \( V \).

A multigraph \((V, A, p)\) is a tuple \((V, A, p)\) where \( V \) and \( A \) are finite sets and \( p: A \rightarrow V \times V \).

Let \( G \) be a directed graph, then \( G \). Directed graphs are multigraphs.
The in-degree of a node in the graph is the number of arcs with target v.

The out-degree of a node in a directed graph is the number of arcs with source v.

Def. Let v be a vertex of a directed graph G.

Simple directed graphs are similar to partial (v, A) with A = V x V.

Allow loops (opps!), whereas we.

Example:
(a) A circuit \((v_0, a_1, v_1, a_2, v_2, \ldots, a_{k-1}, v_k, a_k, v_0)\) is a cycle if \(v_0, v_1, \ldots, v_k\) are distinct, \(k \geq 1\).

(b) A walk from \(v_0\) to \(v_k\) is called a path if \(v_0, v_1, \ldots, v_k\) are distinct.

(c) \(d(v, a) = (v \to a, \infty)\), \(\infty = p, n = 9\).

(d) \(\omega = (a_0, a_1, a_2, \ldots)\), where \(\omega\) is a walk from \(v_0\) to \(v_k\).

(e) \(\omega = (v_0, a_1, v_1, a_2, v_2, \ldots)\) is a multigraph.

(f) \(\omega = (v_0, a_1, v_1, a_2, v_2, \ldots)\) is a multigraph.

(g) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(h) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(i) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(j) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(k) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(l) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(m) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(n) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(o) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(p) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(q) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(r) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(s) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(t) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(u) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(v) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(w) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(x) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(y) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(z) \(\omega = (v_0, a_1, v_1)\) is a multigraph.

(\(\square\))
To Hall's Marriage Theorem, remark that there is an application of digraphs.

For Math 550 notes (lec. 16, Spring 2017) for each vertex $i \in I$ has degree $d_i = d_{-i}$.

Only if each vertex in $E$ has degree $\geq 0$.

3 walk $u \rightarrow v$. Then, $I$ has an Eulerian circuit $I \in$ strongly connected graph. ("Strongly connected" means $Auv$, see

Thm. 2. (Euler - Hierholzer). Let $G$ be a strongly connected

circuit that contains each arc exactly once. $I$ and $G$ to $G$. 13.

In Eulerian circuit in a multigraph is
Proposition.

If \( G \) is not 2-colorable, then \( \chi(G) \geq 3 \).

Examples:

If \( G = \chi(G) = 3 \),

we have \( f(a) \neq f(b) \) for vertices that have the same color  \( (i.e., f(a) \neq f(b)) \).

4-coloring of \( G \) is called proper if no two adjacent

4-color of \( a \) in \( f \),

If \( f \) is a 4-coloring \( \chi(G) \leq 4 \), then \( f(a) \neq f(b) \) whenever the

4-coloring of \( G \) is a map \( \chi \). \( \chi \) is a map \( \chi \). Let \( P \in \Gamma \) be a polynomial.

Let \( G = (V, E, \chi) \) be a multigraph. Let \( P \in \mathbb{N} \),

ON COLORINGS OF GRAPHIC POLYNOMIALS.
Replace 4 by 3.
3-coloring:
To start a proof 4-coloring.

"perfection graph"
Prop 2. Let \( G = (V, E) \) be a simple graph. "Greedy coloring" works: color vertices \( V \) by 4.

Then, \( G \) has a proper \((d+4)\)-coloring.

Let \( d = \max \) degree of a vertex of \( G \),

Prop 1. Let \( G = (V, E') \) be a multigraph.

\[ \text{if } G \text{ has a proper } 2\text{-coloring, then } G \text{ has no odd-length cycles.} \]

\[ \Rightarrow \text{ if } G \text{ has } \# \text{ of component components of } G = 2, \text{ then } \]

\[ \Rightarrow \text{ if } G \text{ has no odd-length circuits.} \]
letters in $G$. $p_n$ is the $n$-th prime (i.e., $\sum_{i=0}^{n} p_i$).

What is $X_e(p)$? $p_i$ is the $i$-th prime. The proper $p$-coloring

$n, n-1, n-2, \ldots$.

Remark: If $p \geq 2$,

$X_e$ is called the chromatic polynomial of $G$.

This $X_e$ is itself.

Let $G = \{v \in V, (v, e) \in E\}$ be a multigraph

$X_e = \sum_{k=0}^{\text{max degree}} X^k(p)$.\hspace{1cm} \text{(Proof: see Exercise 4 on Spring 2017 Math 540 MT 2.)}$

\begin{equation}
X_e = \sum_{k=0}^{\text{max degree}} X^k(p) \hspace{1cm} \text{(Proof: see Exercise 4 on Spring 2017 Math 540 MT 2.)}
\end{equation}

\begin{align*}
\text{For every } G \in \mathcal{G}_n^{\text{proper}}, \quad X_e = \sum_{k=0}^{\text{max degree}} X^k(p).\hspace{1cm} \text{(Proof: see Exercise 4 on Spring 2017 Math 540 MT 2.)}
\end{align*}

Then there exists a unique polynomial $X_e \in \mathbb{Z}[x]$.

Thm. 3. Let $G = (V, E, p)$ be a multigraph.
(a) If \( e \in E \) then \( x_6 \in x_5v \).

(b) If \( x_5v \) is the only edge connecting the endpoints of \( e \),

then \( x_6 = x_5v \). Let \( G \in E \) with \( e \) collapsed.

Let \( G' \in E \) with \( e \) removed.

Let \( G = (V, E', p) \) be a multigraph.

Then, \( G' \) (d/c recurrence).

What is \( x_6 \)?

These are the numbered variables.