Def. If $a \in K[[x]]$ has a mult. inverse $a^{-1}$, then we can define $a^{-n}$ for each $n \in \mathbb{N}$ by setting $a^{-n} = (a^{-1})^n$.

So the powers $a^k$ are defined $\forall k \in \mathbb{Z}$.

Thm. 7. The FPS $1-x$ has a mult. inverse, which is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Proof. $(1-x)(1+x+x^2+x^3 + \ldots) = (1 + x + x^2 + x^3 + \ldots)$

$$- (x + x^2 + x^3 + \ldots) = 1$$

Alternatively:

$$(1-x)(1+x+x^2+x^3 + \ldots)$$

$$= (1-x) + (x-x^2) + (x^2-x^3) + (x^3-x^4) + \ldots = 1$$

Thm. 8. $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \forall n \in \mathbb{Z}$.

(actually an $\infty$ sum if $n < 0$

("Newton's binomial theorem")

Proof idea: for $n \geq 0$, this follows from regular bin. theorem.

For $n = -1$, this boils down to Thm. 7 (with $-x$ replaced by $x$).

For $n = -1$, this boils down to Thm. 7 (with $-x$ replaced by $x$).
Lemma 9. \((1-x)^{-n} = \sum_{k} \binom{n+k-1}{k} x^k\) \(\forall n \in \mathbb{N}\).

Proof Idea for Lemma 9: Induction over \(n\).

Need to check: \(\left( \sum_{k} \binom{n+k-1}{k} x^k \right) \frac{1}{1-x} = \sum_{k} \binom{n+k}{k} x^k\)

\(= \sum_{k} x^k \)

(by Thm.?)

This easily follow from Pascal's recurrence. \(\Box\)

Now, we want to prove Thm. 8 using Lemma 9 by substituting \(-x\) for \(x\). What does "substituting" mean?

We need to define substitution.

**Def 2.** Let \(f, g\) be FPS with \([x^0] g = 0\). (That is, \(g = g_1 x + g_2 x^2 + \ldots\))

**Prop 9.** Let \(f, g\) be FPS with \([x^0] g = 0\). (That is, \(g = g_1 x + g_2 x^2 + \ldots\))

Then, the FPS \(f[Lg]\) is defined as follows:

Write \(f\) as \(f = \sum_{n \geq 0} f_n \cdot x^n\), and set \(f[Lg] = \sum_{n \geq 0} f_n g^n\).
This is well-defined, e.g. the family $(f_n g_n)_{n=0}^\infty$ is summable.

Ex. 2.

We can substitute $x^2 + x^2$ for $x$ into $1 + x + x^2 + \cdots$. The result is

$$1 + \frac{(x+x)^2}{1-x} + \frac{(x+x)^2}{1-x} + \cdots = 1 + x + 2x + 3x^2 + 5x^3 + 8x^4 + \cdots$$

This is because $f_n = \text{Fibonacci} \ #n$.

$$\frac{f_{n+1}}{f_n} = \frac{x + x^2}{1-x} \Rightarrow \text{Ex. } 2.$$

Prop. 10.

The substitution satisfies the rules you'd expect:

\[ f_n \cdot (f \cdot g)^n = f_n \cdot f \cdot (g^n) \]

\[ (f \cdot g)^n \cdot f = f \cdot (g^n \cdot f) \]

(We can verify $T + 2$)

(see below for details)
Thm. 8 (after 2 while).
This justifies Ex. 1.

Rmk. A polynomial is a FPS \((a_0, a_1, \ldots)\) such that all but finitely many \(i \in \mathbb{N}\) satisfy \(a_i = 0\).

To justify Ex. 2, we need to make sense of \((1 + x)^n\) for \(n \in \mathbb{Z}\).

Option 1: Define \((1 + x)^n\) as \(\sum_k \binom{n}{k} x^k\).

But then, we would have to prove all the rules of exponents:

\((1 + x)^n(1 + x)^m = (1 + x)^{n+m}\),

\((1 + x)^m = (1 + x)^m\).

Option 2: Define \((1 + x)^n\) as \(\exp \left[ n \log (1 + x) \right] \).

What are \(\exp\) and \(\log\)?

\(\exp = \sum_{n \geq 0} \frac{1}{n!} x^n\), \quad \log (1 + x) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n\).

Same issues hold, but easier to deal with.
After some work, Ex. 2 becomes justified.

4.3. ONE MORE EXAMPLE FOR FPS

**EX.** \[(1-x)(1+x) = 1-x^2, \]

(\text{Let } n \in \mathbb{N}. \)

\[\Rightarrow \left(1-x\right)^n \left(1+x\right)^n = \left(1-x^2\right)^n\]

\[= \sum_k \left(-1\right)^k \binom{n}{k} x^k = \sum_k \binom{n}{k} x^k = \sum_k \left(-1\right)^k \left(\binom{n}{k}\right) x^{2k}\]

\[\Rightarrow \left(\sum_k \left(-1\right)^k \binom{n}{k} x^k\right) \left(\sum_k \binom{n}{k} x^k\right) = \sum_k \left(-1\right)^k \left(\binom{n}{k}\right) x^{2k}\]

Now take the \text{oeff} \ x^m \text{-coeff. for a given } m \in \mathbb{N},

\[\Rightarrow \sum_{a=0}^{\frac{m}{2}} \left(-1\right)^a \binom{n}{a} \binom{n}{m-a} = \begin{cases} \left(-1\right)^{m/2} \binom{n}{m/2} & \text{if } m \text{ is even} \\
0 & \text{if } m \text{ is odd} \end{cases}\]

For example, if \(m=n\), then this simplifies to

\[\sum_{a=0}^{n} \left(-1\right)^a \binom{n}{a} \binom{n}{n-a} = \begin{cases} \left(-1\right)^{n/2} \binom{n}{n/2} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd} \end{cases}\]

More in [Loehr], [Galvin], etc.
5. Permutations

5.1. Length & Inversions

See [detnotes, Ch. 4].

Recall: A perm. of a set $X$ is a bij. $X \to X$.

Def. Given $n \in \mathbb{N}$, let $S_n$ denote the set of all perms of $[n]$.

This $S_n$ is closed under composition & inverses.

Def. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. The one-line notation of $\sigma$ is the $n$-tuple $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$.

The use of square brackets is a standard; it doesn't mean anything.

Often, the commas and the brackets are omitted.

E.g. the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \in S_5$ is written in one-line notation $[4, 2, 1, 3, 5]$, or short, 42135.
Prop. 1. Let $n \in \mathbb{N}$. Then,

$$S_n \rightarrow [n]_{\text{dist}}^n = \{n\text{-tuples of distinct elts of } [n]\},$$

$$\sigma \rightarrow [\sigma(1), \sigma(2), \ldots, \sigma(n)] = \text{(one-line notation of } \sigma)$$

is a bijection.

(Proof. Pigeonhole principle / box principle / double principle.)

Def. (a) Let $i, j$ be distinct elts of $[n]$. Then, the \textit{transposition} $t_{ij} \in S_n$ is the perm. that sends $i$ to $j$, $j$ to $i$, and leaves everything else in its place.

In one-line notation, $t_{ij} = [2, 3, \ldots, i-1, j, i+1, \ldots, n, i, i+1, \ldots, n]$ for $i < j$.

(b) For each $i \in [n-1]$, set $s_i = t_{i, i+1}$.

Ex. The cycle digraph of $t_{i, j}$ is

$$i \xrightarrow{\circ} j \xrightarrow{\circ} i \xrightarrow{\circ} j \xrightarrow{\circ} \ldots \xrightarrow{\circ} j$$
Prop. 2. (a) \( s_i^2 = \text{id} \quad \forall i. \)

(b) \( s_i \circ s_j = s_j \circ s_i \quad \forall |i-j| > 1. \)

(c) \( s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1} \quad \forall i \in [n-1]. \)

Def. Let \( \omega_0 \) be the pem. in \( S_n \) sending each \( i \) to \( n+1-i \).

In other words, it "reflects" numbers across the center of \( [n] \).

It is the unique strictly decreasing pem. of \( [n] \).

Example. If \( n=5 \), then \( \omega_0 = [5, 4, 3, 2, 1], \)

with cycle digraph \( 5 \xrightarrow{\omega_0} 4 \xrightarrow{\omega_0} 3 \xrightarrow{\omega_0} 2 \xrightarrow{\omega_0} 1 \).

If \( n=6 \), then \( \omega_0 = [6, 5, 4, 3, 2, 1], \)

with cycle digraph \( 6 \xrightarrow{\omega_0} 5 \xrightarrow{\omega_0} 4 \xrightarrow{\omega_0} 3 \xrightarrow{\omega_0} 2 \xrightarrow{\omega_0} 1 \).

Def. For any \( k \) distinct elements \( i_2, \ldots, i_k \) of \( [n] \)

let \( \text{cyc}_{i_2, \ldots, i_k} \) be the pem. in \( S_n \) sending

\( i_1 \rightarrow i_2, \quad i_2 \rightarrow i_3, \ldots, \quad i_{k-1} \rightarrow i_k, \quad i_k \rightarrow i_1, \)
2nd leaving all other numbers in place.

Ex. \[ \text{cyc}(3,5,6) = 3 \longrightarrow 5 \quad \text{cyc} \quad 1 \quad 2 \quad 4 \quad 7 \]

Prop. 3.

(a) \( \text{cyc}_{i_1 \ldots i_k} = t_{i_1, i_2} \circ t_{i_2, i_3} \circ \ldots \circ t_{i_{k-1}, i_k} \)

(b) \( \text{cyc}_{i_1, i_1+1, \ldots, i_1+k-1} = s_{i_1} \circ s_{i_1+1} \circ \ldots \circ s_{i_1+k-2} \)

(c) \( \omega_0 = s_1 \circ (s_2 \circ s_1) \circ (s_3 \circ s_2 \circ s_1) \circ \ldots \circ (s_{n-1} \circ s_{n-2} \circ \ldots \circ s_1) \)

\[ = (s_1 \circ s_2 \circ \ldots \circ s_{n-1}) \circ (s_2 \circ s_1 \circ \ldots \circ s_{n-2}) \circ \ldots \circ (s_1 \circ s_2) \circ s_1 \]

[Here & m the following, \( \alpha \beta \) means \( \alpha \circ \beta \) when \( \alpha, \beta \in S_n \).]

(d) If \( 1 \leq i < j \leq n \), then

\[ t_{i, j} = s_i \circ s_{i+1} \circ s_{j-1} \circ s_{i+1} \circ s_i \]

\[ = s_i \circ s_{i-1} \circ s_i \circ \ldots \circ s_{j-1} \circ s_i \]
(e) $\sigma \circ \text{cyc}_{i_1, \ldots, i_k} \circ^{-1} = \text{cyc}_{\sigma(i_k), \ldots, \sigma(i_1)} \quad \forall \sigma \in S_n.$

(f) $t_{i,j} = \text{cyc}_{i,j}.$

(g) $s_i = \text{cyc}_{i, i+1}.$

Def. An inversion of $\sigma \in S_n$ is a pair $(i,j)$ of elts of $[n]$ with $i < j$ and $\sigma(i) > \sigma(j).$

Example: Let $\tau = [3,1,4,2] \in S_4.$ The inversions of $\tau$ are $(1,2)$ (since $1 < 2$ but $\tau(4) = 3 > 1 = \tau(2)),$ $(1,4),$ and $(3,4).$

Def. The length of $\sigma \in S_n$ is its number of inversions. (Also called the Coxeter length.) It is called $l(\sigma).$

Example: $l(\tau) = 3$ for the $\tau$ above.

Remark: If $\sigma \in S_n,$ then $0 \leq l(\sigma) \leq (n \choose 2).$
The only $\alpha$ with $\ell(\alpha) = 0$ is $\alpha_0$.
The only $\alpha$ with $\ell(\alpha) = (12)$ is $\alpha_2$.

In between, there are many.

In $S_3$, vertices = 6 perms, in $S_2$ in 1-line notation: $l=3 \quad l=2 \quad l=1 \quad l=0$ (or, equivalently, $x = 0 \cdot s_3$ if $x = \alpha_0 s_3$.

We draw an edge $x = \alpha s_i$ if $\ell(x) = i$.