Combinatorics: a first encounter

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1. Preface

These notes (which are work in progress and will remain so for the foreseeable future) are meant as an introduction to combinatorics – the mathematical discipline that studies finite sets (roughly speaking). When finished, they will cover topics such as binomial coefficients, the principles of enumeration, permutations, partitions and graphs. The emphasis falls on enumerative combinatorics, meaning the art of computing sizes of finite sets (“counting”), and graph theory.

I have tried to keep the presentation as self-contained and elementary as possible. The reader is assumed to be familiar with some basics such as induction proofs, equivalence relations and summation signs, as well as have some experience with mathematical proofs. One of the best places to catch up on these basics and to gain said experience is the MIT lecture notes [LeLeMe16] (particularly their first five chapters). Two other resources to familiarize oneself with proofs are [Hammac15] and [Day16]. Generally, most good books about “reading and writing mathematics” or “introductions to abstract mathematics” should convey these skills, although the extent to which they actually do so may differ.

These notes are accompanying two classes on combinatorics (Math 4707 and 4990) I am giving at the University of Minneapolis in Fall 2017.

Here is a (subjective and somewhat random) list of recommended texts on the kinds of combinatorics that will be considered in these notes:
• Enumerative combinatorics (aka counting):
  
  – The very basics of the subject can be found in [LeLeMe16, Chapters 14–15].
  
  – Bogart’s [Bogart04] is an introductory text to enumerative combinatorics that presents the material as an elaborate series of exercises. (Hints and solutions are available from the “Guided Discovery Archive” on the same website.)
  
  – Loehr’s [Loehr11] is a comprehensive textbook on enumerative combinatorics, with a focus on bijections. It goes far beyond these notes, but its first few chapters overlap with what we will do below.
  
  – Galvin’s [Galvin17] contains a readable introduction into combinatorics, including enumeration (but also various more exotic subjects).
  
  – Cameron’s notes [Camero16], [Camero07], [Camero13] (these have significant overlap) are a neat terse introduction into the subject.
  
  – Stanley’s [Stanle12] and [Stanle01] are the most famous treatise on enumerative combinatorics. They are written for graduate students and researchers, so they are not meant as a first introduction; the writing is terse and the exercises often taken from research literature (even following the proofs is challenging at times). We will at most graze this text in the notes below, but you cannot go wrong taking a look at it to see how research-level counting looks like.

• Binomial coefficients:
  
  – A wonderful introduction to this subject is Graham’s, Knuth’s and Patashnik’s [GrKnPa, Chapter 5].
  
  – I give a slow and detailed introduction to binomial coefficients in [Grinbe16, Chapter 2]. Beware that this doesn’t cover half of what we will do in class.
  
  – There are whole books (e.g., [Riorda68]) devoted to combinatorial identities (i.e., identities for binomial coefficients and similar kinds of numbers).

• Generating functions:
  
  – A quick introduction to generating functions, with a few sample applications, appears in [LeLeMe16, Chapter 16].
  
  – The treatment of generating functions in [Loehr11, Chapters 7–8] is probably one of the best in the literature. (It uses concepts from abstract algebra, but these are introduced earlier on in the book.)
  
  – Another introduction is [GrKnPa, Chapter 7].
  
  – Bogart’s [Bogart04] has a chapter on generating functions.
• Permutations:
  - Part of Loehr’s [Loehr11] Chapter 9 is about permutations and determinants.
  - Zeilberger’s paper [Zeilbe85] shows some examples of how properties of determinants can be proven using combinatorial arguments.
  - In [Grinbe16] Chapters 4–5, I cover the theory of permutations necessary to study determinants. Again, I give lots of detail but stay in shallow waters.

• Partitions:
  - Andrews’s and Eriksson’s [AndEri04] is all about this, and supposedly written for undergraduates.
  - For a quick introduction, see [Stanle12] §1.8.
  - Pak’s survey [Pak02] is short on proofs, but gives a deep overview of the subject.
  - Then there is [Wilf09].

• Graphs:
  - [LeLeMe16] Chapter 12] treats the very basics of graph theory.
  - Guichard’s [Guicha16] Chapters 4–5 is another introduction to graph theory.
  - Ore’s [Ore90] is a classical (somewhat informal) introduction meant for laymen. A more rigorous textbook by Ore (that also goes deeper into the subject) is [Ore74].
  - Diestel’s [Dieste16] is a modern graduate textbook on graph theory. It makes no easy reading, but it is rich in content, and I recommend browsing it to get an idea of deep and recent results in graph theory.

• Digraphs:
  - [LeLeMe16] Chapter 10] treats the very basics of digraph (= directed graph) theory.

• Catalan combinatorics:
  - Stanley’s book [Stanle15] is dedicated entirely to Catalan numbers and the various combinatorial objects they count. (Much like [Stanle12], this is a challenging read, mainly meant for graduate students.)

• Pólya enumeration:
  - Loehr’s [Loehr11] Chapter 9 is partly about this.
• Bogart’s [Bogart04, Chapter 6] is devoted to this.

• Combinatorics on words:
  – There are various texts on this; most famous is [Lothai97].

There are numerous other texts. I have heard people recommend [Aigner07] and [Bona11], for example, but I have no experience with these texts myself. I also have learnt that Brualdi’s [Bruald10] has a lot of intersection with what we will do, but I don’t have much first-hands experience with that book.

TODO 1.0.1. List other sources on enumerative combinatorics (use my M.SE post) and graph theory (from my 5707 notes).

The notes you are reading are under construction, and will remain so for at least the whole Fall of 2017. Please let me know of any errors and unclarities you encounter (my email address is dgrinber@umn.edu). Thank you!

1.1. Acknowledgments

Thanks to Angela Chen for corrections.

[Your name could be in here!]

2. What is combinatorics?

This first chapter is meant to give a taste of combinatorics. Rather than systematically study any specific concept, we shall here [...]

2.1. Notations and conventions

Before we get to anything interesting, let us get some technicalities out of the way. Namely, we shall be using the following notations:

• In the following, we use the symbol \( \mathbb{N} \) to denote the set \( \{0, 1, 2, \ldots\} \). (Be warned that some other authors use this symbol for \( \{1, 2, 3, \ldots\} \) instead.)

• We let \( \mathbb{Q} \) denote the set of all rational numbers; we let \( \mathbb{R} \) be the set of all real numbers.

• If \( X \) and \( Y \) are two sets, then we shall use the notation “\( X \rightarrow Y, x \mapsto E \)” (where \( x \) is some symbol which has no specific meaning in the current context, and where \( E \) is some expression which usually involves \( x \)) for “the map from \( X \) to \( Y \) that sends every \( x \in X \) to \( E \)”. For example, “\( \mathbb{N} \rightarrow \mathbb{N}, x \mapsto x^2 + x + 6 \)” means the map from \( \mathbb{N} \) to \( \mathbb{N} \) that sends every \( x \in \mathbb{N} \) to \( x^2 + x + 6 \). For
another example, “\( \mathbb{N} \to \mathbb{Q}, x \mapsto \frac{x}{1+x} \)” denotes the map from \( \mathbb{N} \) to \( \mathbb{Q} \) that sends every \( x \in \mathbb{N} \) to \( \frac{x}{1+x} \).\(^1\)

- If \( S \) is a set, then the power set of \( S \) means the set of all subsets of \( S \). This power set will be denoted by \( \mathcal{P}(S) \). For example, the power set of \( \{1,2\} \) is \( \mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \).

Furthermore, if \( S \) is a set and \( k \) is an integer, then \( \mathcal{P}_k(S) \) shall mean the set of all \( k \)-element subsets of \( S \). (This is empty if \( k < 0 \).)

- If \( f : X \to Y \) is a map, then the following notations are being used:
  - For any subset \( U \) of \( X \), we let \( f(U) \) be the subset \( \{ f(u) \mid u \in U \} \) of \( Y \). This \( f(U) \) is called the image of \( U \) under \( f \).
  - For any subset \( V \) of \( Y \), we let \( f^{-1}(V) \) be the subset \( \{ u \in X \mid f(u) \in V \} \) of \( X \). This \( f^{-1}(V) \) is called the preimage of \( V \) under \( f \).

(Note that in general, \( f(f^{-1}(V)) \neq V \) and \( f^{-1}(f(U)) \neq U \). However, \( f(f^{-1}(V)) \subseteq V \) and \( U \subseteq f^{-1}(f(U)) \).

- An inverse of \( f \) means a map \( g : Y \to X \) satisfying \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \). It is easy to see that an inverse of \( f \) is unique if it exists; it is thus called the inverse of \( f \), and is denoted by \( f^{-1} \). The map \( f \) is said to be invertible if it has an inverse.

- The map \( f \) is said to be injective (or one-to-one) if it has the following property: Whenever two elements \( x_1 \) and \( x_2 \) of \( X \) satisfy \( f(x_1) = f(x_2) \), we must have \( x_1 = x_2 \). (In other words, no two distinct elements of \( X \) are mapped to the same element of \( Y \) by \( f \). In other words, any element \( x \) of \( X \) can be uniquely recovered from its image \( f(x) \).)

- The map \( f \) is said to be surjective (or onto) if \( Y = f(X) \). (Equivalently, \( f : X \to Y \) is surjective if and only if each element of \( Y \) can be written as \( f(x) \) for some \( x \in X \).

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\(^1\)A word of warning: Of course, the notation “\( \mathbb{N} \to \mathbb{Q}, x \mapsto \frac{x}{1+x} \)” does not always make sense; indeed, the map that it stands for might sometimes not exist. For instance, the notation “\( \mathbb{N} \to \mathbb{Q}, x \mapsto \frac{x}{1-x} \)” does not actually define a map, because the map that it is supposed to define (i.e., the map from \( \mathbb{N} \) to \( \mathbb{Q} \) that sends every \( x \in \mathbb{N} \) to \( \frac{x}{1-x} \)) does not exist (since \( \frac{x}{1-x} \) is not defined for \( x = 1 \)). For another example, the notation “\( \mathbb{N} \to \mathbb{Z}, x \mapsto \frac{x}{1+x} \)” does not define a map, because the map that it is supposed to define (i.e., the map from \( \mathbb{N} \) to \( \mathbb{Z} \) that sends every \( x \in \mathbb{N} \) to \( \frac{x}{1+x} \)) does not exist (for \( x = 2 \), we have \( \frac{x}{1+x} = \frac{2}{1+2} \notin \mathbb{Z} \), which shows that a map from \( \mathbb{N} \) to \( \mathbb{Z} \) cannot send this \( x \) to this \( \frac{x}{1+x} \)). Thus, when defining a map from \( X \) to \( Y \) (using whatever notation), do not forget to check that it is well-defined (i.e., that your definition specifies precisely one image for each \( x \in X \), and that these images all lie in \( Y \)). In many cases, this is obvious or very easy to check (I will usually not even mention this check), but in some cases, this is a difficult task.
The map $f$ is said to be bijective (or a one-to-one correspondence) if it is both injective and surjective. Equivalently, a map $f : X \to Y$ is bijective if and only if it is invertible (i.e., has an inverse).

3. The twelve-fold way

References


Notes on graph theory (Monday 2nd October, 2017, 8:02pm)

http://www.people.vcu.edu/~rhammack/BookOfProof/


http://users.utu.fi/harju/graphtheory/graphtheory.pdf


https://www.quantamagazine.org/20170105-graph-isomorphism-retraction/


[Petrov15] Fedor Petrov, mathoverflow post #198679 (Flooding a cycle digraph via chip-firing: \( n^{k-1} + n^{k-2} + \cdots + 1 \) bound (a Norway 1998-99 problem generalized)), MathOverflow, http://mathoverflow.net/q/198679


See http://math.mit.edu/~rstan/ec/ for errata and addenda.

See http://math.mit.edu/~rstan/catalan/ for errata and addenda.


https://neil-strickland.staff.shef.ac.uk/courses/MAS201/


See http://www.math.illinois.edu/~dwest/igt/ for updates and corrections.

https://www.math.upenn.edu/~wilf/PIMS/PIMSLectures.pdf

See https://archive.org/details/IntroductionToGraphTheory for the 3rd edition.