Definition 0.1. Let $A$ be a logical statement. Then, an element $[A] \in \{0, 1\}$ is defined as follows: We set $[A] = \begin{cases} 1, & \text{if } A \text{ is true;} \\ 0, & \text{if } A \text{ is false} \end{cases}$. This element $[A]$ is called the truth value of $A$. (For example, $[1 + 1 = 2] = 1$ and $[1 + 1 = 3] = 0$.) The notation $[A]$ for the truth value of $A$ is known as the Iverson bracket notation.

Exercise 1. Prove the following rules of truth values:

(a) If $A$ and $B$ are two equivalent logical statements, then $[A] = [B]$.

(b) If $A$ is any logical statement, then $[\neg A] = 1 - [A]$.

(c) If $A$ and $B$ are two logical statements, then $[A \land B] = [A] \cdot [B]$.

(d) If $A$ and $B$ are two logical statements, then $[A \lor B] = [A] + [B] - [A] \cdot [B]$.

(e) If $A$, $B$, and $C$ are three logical statements, then 

$$[A \lor B \lor C] = [A] + [B] + [C] - [A] \cdot [B] - [A] \cdot [C] - [B] \cdot [C] + [A] \cdot [B] \cdot [C].$$

Definition 0.2. We define the binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}$$

for every $n \in \mathbb{Q}$ and $k \in \mathbb{N}$. (Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$, and that an empty product is defined to be 1.)

For example, $\binom{-3}{4} = \frac{(-3)(-4)(-5)(-6)}{4!} = 15$ and $\binom{4}{1} = \frac{4}{1!} = 4$ and $\binom{4}{0} = \frac{\text{(empty product)}}{0!} = \frac{1}{1} = 1$.

Exercise 2. Prove the following:

(a) We have $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ for any $n \in \mathbb{Q}$ and $k \in \mathbb{N}$.

(b) We have $k \binom{n}{k} = n \binom{n-1}{k-1}$ for any $n \in \mathbb{Q}$ and any positive integer $k$.

(c) If $n \in \mathbb{Q}$ and if $a$ and $b$ are two integers such that $a \geq b \geq 0$, then

$$\binom{n}{a} \binom{a}{b} = \binom{n}{b} \binom{n-b}{a-b}.$$
[Caveat: You may have seen the formula \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \). But this formula only makes sense when \( n \) and \( k \) are nonnegative integers and \( n \geq k \). Thus it is not general enough to be used in this exercise.]

**Exercise 3.** Let \( k \) be a positive integer.

(a) How many \( k \)-digit numbers are there? (A “\( k \)-digit number” must have \( k \) digits without leading zeroes. For example, 3902 is a 4-digit number, not a 5-digit number. Note that 0 counts as a 0-digit number, not as a 1-digit number.)

(b) How many \( k \)-digit numbers are there that have no two equal digits?

(c) How many \( k \)-digit numbers have an even sum of digits?

(d) How many \( k \)-digit numbers are palindromes? (A “palindrome” is a number such that reading its digits from right to left yields the same number. For example, 5, 1331 and 49094 are palindromes. Your answer may well depend on the parity of \( k \).)

For each \( n \in \mathbb{N} \), we set \([n] = \{1, 2, \ldots, n\}\).

**Definition 0.3.** The [Fibonacci sequence](https://en.wikipedia.org/wiki/Fibonacci_number) is the sequence \((f_0, f_1, f_2, \ldots)\) of integers which is defined recursively by \( f_0 = 0, f_1 = 1, \) and \( f_n = f_{n-1} + f_{n-2} \) for all \( n \geq 2 \). Its first terms are

\[
\begin{align*}
  f_0 &= 0, & f_1 &= 1, & f_2 &= 1, & f_3 &= 2, & f_4 &= 3, & f_5 &= 5, \\
  f_6 &= 8, & f_7 &= 13, & f_8 &= 21, & f_9 &= 34, & f_{10} &= 55, \\
  f_{11} &= 89, & f_{12} &= 144, & f_{13} &= 233.
\end{align*}
\]

(Some authors prefer to start the sequence at \( f_1 \) rather than \( f_0 \); of course, the recursive definition then needs to be modified to require \( f_2 = 1 \) instead of \( f_0 = 0 \).)

**Exercise 4.** A set \( S \) of integers is said to be lacunar if no two consecutive integers occur in \( S \) (that is, there exists no \( i \in \mathbb{Z} \) such that both \( i \) and \( i+1 \) belong to \( S \)). For example, \( \{1,3,6\} \) is lacunar, but \( \{2,4,5\} \) is not. (The empty set and any 1-element set are lacunar, of course.)

For a positive integer \( n \), let \( g(n) \) denote the number of all lacunar subsets of \([n]\).

(a) Compute \( g(n) \) for all \( n \in \{1,2,3,4,5\} \).

(b) Find and prove a recursive formula for \( g(n) \) in terms of \( g(n-1) \) and \( g(n-2) \).

(c) Prove that \( g(n) = f_{n+2} \) for each \( n \in \mathbb{N} \).

Recall that if \( a, b \) and \( m \) are three integers (with \( m > 0 \)), then we write \( a \equiv b \mod m \) if and only if \( a - b \) is divisible by \( m \). Thus, in particular, \( a \equiv b \mod 2 \) if and only if \( a \) and \( b \) have the same parity (i.e., are either both even or both odd).
Exercise 5. A set $S$ of integers is said to be $O<E<O<E<\ldots$ (this is an adjective) if it can be written in the form $S = \{s_1, s_2, \ldots, s_k\}$ where

- $s_1 < s_2 < \cdots < s_k$;
- the integer $s_i$ is even whenever $i$ is even;
- the integer $s_i$ is odd whenever $i$ is odd.

(For example, $\{1, 4, 5, 8, 11\}$ is an $O<E<O<E<\ldots$ set, while $\{2, 3\}$ and $\{1, 4, 6\}$ are not. Note that $k$ is allowed to be 0, whence $\emptyset$ is an $O<E<O<E<\ldots$ set.)

For each $n \in \mathbb{N}$, we let $a(n)$ denote the number of all $O<E<O<E<\ldots$ subsets of $[n]$, and let $b(n)$ denote the number of all $O<E<O<E<\ldots$ subsets of $[n]$ that contain $n$.

(a) Show that $a(n) = a(n - 1) + b(n)$ for each $n > 0$.

(b) Show that $a(n) = 1 + \sum_{k=0}^{n} b(k)$ for each $n \in \mathbb{N}$.

(c) Show that $b(n) = \sum_{k \in \{0, 1, \ldots, n-1\} \mid k \equiv n-1 \pmod{2}} b(k) + [n \text{ is odd}]$ for each $n \in \mathbb{N}$.

(d) Show that $b(n) + b(n - 1) = 1 + \sum_{k=0}^{n-1} b(k)$ for each $n > 0$.

(e) Show that $b(n) = 1 + \sum_{k=0}^{n-2} b(k)$ for each $n > 0$.

(f) Show that $b(n) = a(n - 2)$ for each $n \geq 2$.

(g) Show that $a(n) = f_{n+2}$ for each $n \in \mathbb{N}$.

[Hint: You may skip parts (b)–(e) if you can prove part (f) without using any of them.]

Remark 0.4. Comparing Exercise 4 (c) with Exercise 5 (g) tells us that there are precisely as many lacunar subsets of $[n]$ as there are $O<E<O<E<\ldots$ subsets of $[n]$. Is there a bijection between the former and the latter? I don’t know.

Exercise 6. For each $n \in \mathbb{N}$, we let $c(n)$ denote the number of all subsets of $[n]$ that are simultaneously lacunar and $O<E<O<E\ldots$.

Prove that $c(n) = c(n - 2) + c(n - 3)$ for all $n \geq 3$.

Remark 0.5. The sequence $(c(1), c(2), c(3), \ldots)$ from Exercise 6 is the Padovan sequence (starting with 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49).

Exercise 7. Extend the “twelvefold way” by a new column: counting only the bijective maps $f : \mathbb{N} \to K$. Fill in this column (all of its four boxes).
Appendix

Here is a sample exercise (no points for this one...) with a solution. This should give you some idea of what level of detail I expect in your solutions.

**Exercise 8.** A set $S$ of integers is said to be *self-counting* if the size $|S|$ is an element of $S$. (For example, $\{1, 3, 5\}$ is self-counting, since $|\{1, 3, 5\}| = 3 \in \{1, 3, 5\}$; but $\{1, 2, 5\}$ is not self-counting.)

Let $n$ be a positive integer.

(a) For each $k \in [n]$, show that the number of self-counting subsets of $[n]$ having size $k$ is $\binom{n-1}{k-1}$.

(b) Conclude that the number of self-counting subsets of $[n]$ is $\sum_{k=0}^{n-1} \binom{n-1}{k}$.

(c) Find and prove a simpler expression for this number.

**Solution to Exercise 8.** (a) Fix $k \in [n]$. Then, the self-counting subsets of $[n]$ having size $k$ are exactly the subsets of $[n]$ having size $k$ and containing $k$. Thus, the maps

$$\{\text{self-counting subsets of } [n] \text{ having size } k\} \to \{\text{subsets of } [n] \setminus \{k\} \text{ having size } k - 1\},$$

$$S \mapsto S \setminus \{k\}$$

and

$$\{\text{subsets of } [n] \setminus \{k\} \text{ having size } k - 1\} \to \{\text{self-counting subsets of } [n] \text{ having size } k\},$$

$$S \mapsto S \cup \{k\}$$
are well-defined\(^1\) mutually inverse\(^2\) and thus are bijections. Hence,

\[
\left|\{\text{self-counting subsets of } [n] \text{ having size } k\}\right| = \left|\{\text{subsets of } [n] \setminus \{k\} \text{ having size } k - 1\}\right| = \binom{|[n] \setminus \{k\}|}{k - 1}
\]

because for any finite set \(Q\) and any \(m \in \mathbb{N}\), we have

\[
\left|\{\text{subsets of } Q \text{ having size } m\}\right| = \binom{|Q|}{m}
\]

\[
= \binom{n - 1}{k - 1} \quad \text{(since } |[n] \setminus \{k\}| = n - 1).\]

This proves part \((a)\).

\((b)\) Any self-counting subset of \([n]\) must have at least one element (namely, its size); thus, its size must be one of the integers \(1, 2, \ldots, n\). Hence,

\[
\left|\{\text{self-counting subsets of } [n]\}\right| = \sum_{k=1}^{n} \left|\{\text{self-counting subsets of } [n] \text{ having size } k\}\right| = \binom{n - 1}{k - 1} \quad \text{(by part } (a)\text{)}
\]

\[
= \sum_{k=1}^{n} \binom{n - 1}{k - 1} = \sum_{k=0}^{n-1} \binom{n - 1}{k}
\]

(here, we have substituted \(k\) for \(k - 1\) in the sum). This proves part \((b)\).

\(^1\)This means the following:

- If \(S\) is a self-counting subset of \([n]\) having size \(k\), then \(S \setminus \{k\}\) is a subset of \([n] \setminus \{k\}\) having size \(k - 1\).
- If \(S\) is a subset of \([n] \setminus \{k\}\) having size \(k - 1\), then \(S \cup \{k\}\) is a self-counting subset of \([n]\) having size \(k\).

Checking this is straightforward; you can do it in your head, but don’t forget to do this! If you don’t check well-definedness, then it may happen that one of your “maps” does not exist; for example, convince yourself that there is no map

\[
\{\text{subsets of } [n]\} \rightarrow \{\text{subsets of } [n]\},
\]

\[
S \mapsto S \cup \{|S| + 1\},
\]

because the set \(S \cup \{|S| + 1\}\) is not always a subset of \([n]\) (namely, it fails to be so when \(|S| = n\)).

\(^2\)For this, you need to show that

- If \(S\) is a self-counting subset of \([n]\) having size \(k\), then \((S \setminus \{k\}) \cup \{k\} = S\).
- If \(S\) is a subset of \([n] \setminus \{k\}\) having size \(k - 1\), then \((S \cup \{k\}) \setminus \{k\} = S\).

This is again entirely straightforward, and it is perfectly fine to do this in your head, but you should do it.
(c) This number is $2^{n-1}$.

Proof. In light of part (b), it suffices to show that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}. \quad (1)$$

In order to do so, it suffices to prove the identity

$$\sum_{k=0}^{m} \binom{m}{k} = 2^{m} \quad \text{for all } m \in \mathbb{N} \quad (2)$$

(because we can then apply (2) to $m = n - 1$, and obtain (1)).

The identity (2) is well-known (it says that the sum of all entries in the $m$-th row of Pascal’s triangle is $2^m$), but let us sketch a quick combinatorial proof: The number of all subsets of $[m]$ is $2^m$ (because to choose such a subset means to decide, for each element of $[m]$, whether it goes into the subset or not; thus, we have 2 choices for each element, and $m$ elements, whence there is a total of $2^m$ possibilities). On the other hand, this number equals

$$\sum_{k=0}^{m} \binom{m}{k} \quad \text{(the number of all } k\text{-element subsets of } [m]) = \sum_{k=0}^{m} \binom{m}{k}. \quad (3)$$

Comparing the two results, we obtain $\sum_{k=0}^{m} \binom{m}{k} = 2^m$ (because both results are the same number – viz., the number of all subsets of $[m]$). Thus, (2) is proven, and the proof of part (c) is thus complete. \qed