0.1. Counting triples

Recall that the word "triple" means a 3-tuple. Tuples are always ordered by definition.

Exercise 1. Let \( n \in \mathbb{N} \).

(a) Find the number of all triples \((A, B, C)\) of subsets of \([n]\) satisfying \(A \cup B \cup C = [n]\) and \(A \cap B \cap C = \emptyset\).

(b) Find the number of all triples \((A, B, C)\) of subsets of \([n]\) satisfying \(B \cap C = C \cap A = A \cap B\).

(c) Find the number of all triples \((A, B, C)\) of subsets of \([n]\) satisfying \(A \cap B = A \cap C\).

Solution to Exercise 1 (sketched). (a) The number of such triples is \(6^n\).

Proof. Let me first give a quick but informal argument.

Clearly, a triple \((A, B, C)\) of subsets of \([n]\) satisfies \(A \cup B \cup C = [n]\) and \(A \cap B \cap C = \emptyset\) if and only if it has the following property: Each \(i \in [n]\) belongs to at least one of the three sets \(A, B, C\), but no \(i \in [n]\) belongs to all three of them. Thus, the following simple algorithm constructs every triple \((A, B, C)\) of subsets of \([n]\) satisfying \(A \cup B \cup C = [n]\) and \(A \cap B \cap C = \emptyset\): For each \(i \in [n]\), we decide whether the element \(i\) should be contained in the set \(A\) only (i.e., in \(A\) but not in \(B\) and not in \(C\)), or in the set \(B\) only, or in the set \(C\) only, or in the sets \(A\) and \(B\) only (i.e., in \(A\) and \(B\) but not in \(C\)), or in the sets \(A\) and \(C\) only, or in the sets \(B\) and \(C\) only. There are clearly 6 options to choose from in this decision. Thus, in total, there are \(6^n\) possible triples (because we are making this decision once for each of the \(n\) elements \(i\) of \([n]\)). This completes our informal proof.

A rigorous way to present the above argument is the following: Let \(\mathcal{A}\) be the set of all triples \((A, B, C)\) of subsets of \([n]\) satisfying \(A \cup B \cup C = [n]\) and \(A \cap B \cap C = \emptyset\). We must show that \(|\mathcal{A}| = 6^n\). We know that the set \([6]^{[n]}\) (that is, the set of all maps \([n] \to [6]\)) has size \(|[6]^{[n]}| = |[6]|^{|[n]|} = 6^n\); thus, it will suffice to exhibit a bijection \(\mathcal{A} \to [6]^{[n]}\).

We define such a bijection \(\Sigma : \mathcal{A} \to [6]^{[n]}\) as follows: It should send any triple \((A, B, C) \in \mathcal{A}\) to the map \(f : [n] \to [6]\) that sends each \(i \in [n]\) to

\[
\begin{align*}
1, & \quad \text{if } i \in A \text{ but } i \notin B \text{ and } i \notin C; \\
2, & \quad \text{if } i \in B \text{ but } i \notin C \text{ and } i \notin A; \\
3, & \quad \text{if } i \in C \text{ but } i \notin A \text{ and } i \notin B; \\
4, & \quad \text{if } i \in A \text{ and } i \in B \text{ but } i \notin C; \\
5, & \quad \text{if } i \in A \text{ and } i \in C \text{ but } i \notin B; \\
6, & \quad \text{if } i \in B \text{ and } i \in C \text{ but } i \notin A.
\end{align*}
\]
is well-defined for each construct an inverse for \((\text{the triple } \Xi)\). Indeed, it is straightforward to check that this map is well-defined, and actually so I was looking for a map that reconstructs any triple \((A, B, C)\) of subsets of \(\mathcal{A}\) from the map \(f : [n] \to [6]\) that sends each \(i \in [n]\) to \(f(i)\). This is a rather simple reconstruction problem: For example, the first entry \(A\) of this triple \((A, B, C)\) can be reconstructed from \(f\) as the set \(\{i \in [n] \mid f(i) \in \{1, 4, 5\}\}\), because the elements of \(A\) are exactly those elements \(i \in [n]\) whose image under \(f\) is 1, 4 or 5.) This completes the rigorous proof of (a).

(b) The number of such triples is \(5^n\).

Proof. I shall give an informal proof only, trusting that you can translate it into a rigorous bijective argument as I’ve done above for part (a).

Clearly, a triple \((A, B, C)\) of subsets of \([n]\) satisfies \(B \cap C = C \cap A = A \cap B\) if and only if it has the following property: Each \(i \in [n]\) either belongs to at most one of the three sets \(A, B, C\), or belongs to all three of them. Thus, the following simple algorithm constructs every triple \((A, B, C)\) of subsets of \([n]\) satisfying \(B \cap C = C \cap A = A \cap B\): For each \(i \in [n]\), we decide whether the element \(i\) should be contained in none of the sets \(A, B, C\), or in the set \(A\) only (i.e., in \(A\) but not in \(B\) and not in \(C\)), or in the set \(B\) only, or in the set \(C\) only, or in all three sets \(A, B\) and

\[
\begin{align*}
A &= \{i \in [n] \mid f(i) \in \{1, 4, 5\}\}; \\
B &= \{i \in [n] \mid f(i) \in \{2, 4, 6\}\}; \\
C &= \{i \in [n] \mid f(i) \in \{3, 5, 6\}\}.
\end{align*}
\]

Indeed, it is straightforward to check that this map is well-defined, and actually inverse to \(\Xi\). (How did I come up with this map? Well, I wanted an inverse to \(\Xi\) that sends each \(f : [n] \to [6]\) to the triple \((A, B, C)\), where

1From the point of view of logic, the word “but” is merely a synonym for “and”. But in this definition, it is meant to reinforce the intuition: We say “if \(i \in A\) but \(i \notin B\) and \(i \notin C\)” because we clearly want to contrast the sets to which \(i\) belongs on one side against the sets to which \(i\) does not belong on the other.

2It is clear enough that \(i\) cannot satisfy more than one of these conditions. In order to see that \(i\) has to satisfy at least one of them, we must rule out the possibilities that \((i \in A \text{ and } i \in B \text{ and } i \in C)\) and \((i \notin A \text{ and } i \notin B \text{ and } i \notin C)\). But this is easy: The first of these possibilities is ruled out by \(A \cap B \cap C = \emptyset\), while the second is ruled out by \(A \cup B \cup C = [n]\).
C. There are clearly 5 options to choose from in this decision. Thus, in total, there are $5^n$ possible triples (because we are making this decision once for each of the $n$ elements $i$ of $[n]$). This completes our informal proof.

(c) The number of such triples is $6^n$.

Proof. I shall give an informal proof only, trusting that you can translate it into a rigorous bijective argument as I’ve done above for part (a).

Clearly, a triple $(A, B, C)$ of subsets of $[n]$ satisfies $B \cap C = C \cap A = A \cap B$ if and only if it has the following property: Each $i \in [n]$ either belongs to at most one of the three sets $A$, $B$, and $C$, or belongs to $B$ and $C$ only, or belongs to all three of them. Thus, the following simple algorithm constructs every triple $(A, B, C)$ of subsets of $[n]$ satisfying $A \cap B = A \cap C$: For each $i \in [n]$, we decide whether the element $i$ should be contained in none of the sets $A$, $B$, and $C$, or in the set $A$ only (i.e., in $A$ but not in $B$ and not in $C$), or in the set $B$ only, or in the set $C$ only, or in the sets $B$ and $C$ only (i.e., in $B$ and in $C$ but not in $A$), or in all three sets $A$, $B$, and $C$. There are clearly 6 options to choose from in this decision. Thus, in total, there are $6^n$ possible triples (because we are making this decision once for each of the $n$ elements $i$ of $[n]$). This completes our informal proof. \qed

0.2. Stirling numbers of the 2nd kind, again

Recall that if $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $\text{sur}(n, k)$ denotes the number of surjections $[n] \to [k]$, and $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the Stirling number of the 2nd kind (defined as $\text{sur}(n, k)/k!$).

Recall furthermore that we are using the convention that $\binom{a}{b} = 0$ when $b \notin \mathbb{N}$.

Exercise 2. Let $n$ be a positive integer. Let $k \in \mathbb{N}$.

(a) Prove that

$$\text{sur}(n, k) = k \sum_{i=0}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^{n-1}.$$ 

(b) Prove that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{i=0}^{k} (-1)^{k-i} \frac{i^n}{i!(k-i)!}.$$ 

Solution to Exercise 2. Exercise 4 on Math 4990 homework set #2 showed that

$$\text{sur}(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n. \quad (2)$$

But Exercise 2 (b) on Math 4990 homework set #1 showed that

$$K \binom{N}{K} = N \binom{N-1}{K-1} \quad (3)$$
for any $N \in \mathbb{Q}$ and any positive integer $K$. (The variables $N$ and $K$ in this equality have been called $n$ and $k$ in the exercise we have cited, but we are using the notations $n$ and $k$ for different purposes here.) Furthermore, we know that

$$
\binom{N}{K} = \frac{N!}{K!(N-K)!}
$$
for each $N \in \mathbb{N}$ and each $K \in \{0, 1, \ldots, N\}$.

(a) From (2), we obtain

$$
sur(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n = (-1)^{k-0} \binom{k}{0} + \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n
$$

(since $n$ is positive)

$$
= (-1)^{k-0} \binom{k}{0} + \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n = \sum_{i=1}^{k} (-1)^{k-i} \left( \binom{k}{i} i^n \right) + \sum_{i=1}^{k} (-1)^{k-i} \left( \binom{k}{i} i^n \right)
$$

(since $n$ is positive)

$$
= \sum_{i=1}^{k} (-1)^{k-i} \left( \binom{k}{i} i^n \right) = \sum_{i=1}^{k} \left( -1 \right)^{k-i} \left( \binom{k}{i} i^n \right)
$$

(by (3), applied to $N=k$ and $K=i$)

$$
= \sum_{i=1}^{k} (-1)^{k-i} k \left( \binom{k-1}{i-1} i^n \right) = k \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^n - 1.
$$

Comparing this with

$$
k \left( \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n - 1 \right)
$$

$$
= (-1)^{k-0} \binom{k-1}{0-1} 0^n + \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^n - 1
$$

(since $0-1<0$)

$$
= k \left( \binom{k-1}{0-1} 0^n + \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^n - 1 \right)
$$

$$
= k \left( (-1)^{k-0} \binom{k-1}{0-1} 0^n + \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^n - 1 \right)
$$

$$
= k \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^n - 1,
$$
we obtain

\[ \text{sur} \left( n, k \right) = k \sum_{i=0}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^{n-1}. \]

This solves Exercise 2(a).

(b) We know from class that

\[ \binom{n}{k} = \text{sur} \left( n, k \right) \]

\[ = \frac{k!}{i! (k-i)!} \]

(by (2))

\[ = \frac{1}{i! (k-i)!} \cdot \frac{k!}{i! (k-i)!} \]

(by (1), applied to \( N = k \) and \( K = i \))

\[ = \sum_{i=0}^{k} (-1)^{k-i} \frac{1}{i! (k-i)!} i^n = \sum_{i=0}^{k} (-1)^{k-i} \frac{i^n}{i! (k-i)!}. \]

This solves Exercise 2(b). \( \square \)

### 0.3. Counting 2-lacunar subsets

**Exercise 3.** A set \( S \) of integers is said to be 2-lacunar if every \( i \in S \) satisfies \( i+1 \notin S \) and \( i+2 \notin S \). (That is, any two distinct elements of \( S \) are at least a distance of 3 apart on the real axis.) For example, \( \{1, 5, 8\} \) is 2-lacunar, but \( \{1, 5, 7\} \) is not.

For any \( n \in \mathbb{N} \), we let \( h(n) \) denote the number of all 2-lacunar subsets of \( [n] \).

(a) Prove that \( h(n) = h(n-1) + h(n-3) \) for each \( n \geq 3 \).

(b) Prove that \( h(n) = \sum_{k \in \mathbb{N}, 2k \leq n+2} \binom{n+2-2k}{k} \) for each \( n \in \mathbb{N} \).

**Solution to Exercise 3 (sketched).** Most of the arguments used in this exercise are straightforward adaptations of arguments used in Exercise 4(b) on Math 4990 homework set #1 and in Exercise 3 on Math 4990 homework set #2. Thus, we shall be very brief this time, pointing out only the differences.

(a) Exercise 3 is solved in the same way as Exercise 4(b) on Math 4990 homework set #1 was solved. This time, of course, instead of finding a bijection from \( \{S \subseteq [n] \mid S \text{ is lacunar and } n \in S \} \) to \( \{S \subseteq [n-2] \mid S \text{ is lacunar} \} \), we need to find a bijection from \( \{S \subseteq [n] \mid S \text{ is 2-lacunar and } n \in S \} \) to \( \{S \subseteq [n-3] \mid S \text{ is 2-lacunar} \} \).

The bijection is defined in exactly the same way as before: It sends each \( T \) to \( T \setminus \{n\} \).

(b) We begin with the following fact:
**Observation 0:** Let $S$ be a 2-lacunar subset of $[n]$. Then,

$$|S| \leq \frac{n+2}{3}.$$ 

**Proof of Observation 0:** Let $S'$ be the subset $\{s + 1 \mid s \in S\}$ of $[n+1]$. Let $S''$ be the subset $\{s + 2 \mid s \in S\}$ of $[n+2]$. Both subsets $S'$ and $S''$ are just copies of $S$, shifted by 1 and by 2, respectively; thus, their sizes are the same as the size of $S$: that is, we have $|S| = |S'| = |S''|$. Also, it is easy to see that the three sets $S, S', S''$ are disjoint. Hence, $|S \cup S' \cup S''| = |S| + |S'| + |S''| = 3|S|$ (since $|S| = |S'| = |S''|$). But $S, S'$ and $S''$ are subsets of $[n+2]$; therefore, so is $S \cup S' \cup S''$. Hence, $|S \cup S' \cup S''| \leq |[n+2]| = n+2$. In view of $|S \cup S' \cup S''| = 3|S|$, this rewrites as $3|S| \leq n+2$, so that $|S| \leq \frac{n+2}{3}$. This proves Observation 0.

Next, we need to prove the following statement:

**Observation 1:** Let $n \in \mathbb{N}$. For any $k \in \mathbb{N}$ satisfying $2k \leq n+2$, the number of all 2-lacunar $k$-element subsets of $[n]$ is $\binom{n-2k+2}{k}$.

**Proof of Observation 1:** One way to prove this is analogous to the first solution of Exercise 3 (a) on [Math 4990 homework set #2](math4990hwset2). The main differences are:

- The set $\text{Lac}_k(n)$ of all lacunar $k$-element subsets of $[n]$ is replaced by the set $\text{Lac}_{k,2}(n)$ of all 2-lacunar $k$-element subsets of $[n]$.
- The maps $\Phi : \text{Lac}_k(n) \to \mathcal{P}_k([n-k+1])$ and $\Psi : \mathcal{P}_k([n-k+1]) \to \text{Lac}_k(n)$ are replaced by maps $\Phi : \text{Lac}_{k,2}(n) \to \mathcal{P}_k([n-2k+2])$ and $\Psi : \mathcal{P}_k([n-2k+2]) \to \text{Lac}_{k,2}(n)$ defined as follows: $\Phi$ sends any $S = \{s_1 < s_2 < \cdots < s_k\} \in \text{Lac}_{k,2}(n)$ to $\{s_1 - 0 < s_2 - 2 < s_3 - 4 < \cdots < s_k - 2(k-1)\} = \{s_i - 2(i-1) \mid i \in [k]\}$, whereas $\overline{\Psi}$ sends any $T = \{t_1 < t_2 < \cdots < t_k\} \in \mathcal{P}_k([n-2k+2])$ to $\{t_1 + 0 < t_2 + 2 < t_3 + 4 < \cdots < t_k + 2(k-1)\} = \{t_i + 2(i-1) \mid i \in [k]\}$.

(In other words, instead of increasing/decreasing gaps between neighboring elements of the subset by 1, we are now increasing/decreasing them by 2.)

---

3Proof. Let us just check that $S$ and $S''$ are disjoint. (The other two statements are proven similarly.)

Indeed, let $j \in S \cap S''$. Then, $j \in S$ and $j \in S''$. From $j \in S''$, it follows that $j = s + 2$ for some $s \in S$ (by the definition of $S''$). Consider this $s$. Now, recall that every $i \in S$ satisfies $i + 2 \notin S$ (since $S$ is lacunar). Applying this to $i = s$, we obtain $s + 2 \notin S$. This contradicts $s + 2 = j \in S$.

Now, forget that we fixed $j$. We thus have obtained a contradiction for each $j \in S \cap S''$. Hence, there exists no $j \in S \cap S''$. In other words, the sets $S$ and $S''$ are disjoint.
Alternatively, Observation 1 can also be proven similarly to the second solution of Exercise 3 on Math 4990 homework set #2. The analogue of Claim 1 should now state that \( g_k(n) = g_k(n-1) + g_{k-1}(n-3) \) for all \( n \geq 1 \) and \( k \in \mathbb{Z} \) (where \( g_k(n) \) denotes the number of all 2-lacunar \( k \)-element subsets of \([n] \)); and the analogue of Claim 2 should now state that each \( n \in \{-2,-1,1,2,\ldots\} \) and \( k \in \mathbb{N} \) with \( 2k \leq n+2 \) satisfy \( g_k(n) = \binom{n-2k+2}{k} \). (In the proof of Claim 2, the case of \( 2k = m + 2 \) needs to be treated separately, in the same way as we had to treat the case \( k = m + 1 \) separately back in homework set #2. This is slightly harder this time, however. Observation 0 shows that a \( k \)-element 2-lacunar subset of \([m] \) must have size \( k \leq \frac{m+2}{3} \) and \( \frac{m+2}{2} \), whence it cannot satisfy \( 2k = m + 2 \) unless \( m = -2 \).) Either way, Observation 1 is eventually proven.

Now, we proceed similarly to the solution of Exercise 3 on Math 4990 homework set #2. Fix \( n \in \mathbb{N} \). The size of any 2-lacunar subset of \([n] \) is a \( k \in \mathbb{N} \) satisfying \( 2k \leq n+2 \) (because Observation 0 yields that it is \( \leq \frac{n+2}{3} \) and \( \leq \frac{n+2}{2} \)). Now,

\[
 h(n) = (\text{the number of all 2-lacunar subsets of } [n])
 = \sum_{k \in \mathbb{N}; \ 2k \leq n+2} (\text{the number of all 2-lacunar subsets of } [n] \text{ having size } k)
 = \left( \text{the number of all 2-lacunar } k\text{-element subsets of } [n] \right)
 = \binom{n-2k+2}{k} \\
(\text{by Observation 1})
\]

because the size of a 2-lacunar subset of \([n] \) is a \( k \in \mathbb{N} \) satisfying \( 2k \leq n+2 \)

\[
 = \sum_{k \in \mathbb{N}; \ 2k \leq n+2} \binom{n-2k+2}{k} = \sum_{k \in \mathbb{N}; \ 2k \leq n+2} \binom{n+2-2k}{k}.
\]

This solves Exercise 3(b).

0.4. Counting shadowed subsets

**Exercise 4.** A set \( S \) of integers is said to be shadowed if it has the following property: Whenever an odd integer \( i \) belongs to \( S \), the next integer \( i+1 \) must also belong to \( S \). (For example, \( \emptyset \), \( \{2,4\} \) and \( \{1,2,5,6,8\} \) are shadowed, but \( \{1,5,6\} \) is not, since 1 belongs to \( \{1,5,6\} \) but 2 does not.)

(a) Let \( n \in \mathbb{N} \) be even. How many shadowed subsets of \([n] \) exist?

(b) Let \( n \in \mathbb{N} \) be odd. How many shadowed subsets of \([n] \) exist?

**Solution to Exercise 4 (sketched).** (a) The number of shadowed subsets of \([n] \) is \( 3^{n/2} \).

**Proof.** Here is an informal argument:
The definition of a “shadowed” set can be rewritten as follows: A set $S$ of integers is shadowed if and only if, for each integer $i$, it either contains none of the two integers $2i - 1$ and $2i$, or it contains $2i$ but not $2i - 1$, or it contains both $2i - 1$ and $2i$. (What it cannot do is contain $2i - 1$ but not $2i$.) When we are studying subsets of $[n]$, we can restrict ourselves to only considering the integers $i \in [n/2]$, because each of the elements of $[n]$ can be uniquely represented in the form $2i - 1$ or in the form $2i$ for some $i \in [n/2]$. Thus, a subset $S$ of $[n]$ is shadowed if and only if, for each $i \in [n/2]$, it either contains none of the two integers $2i - 1$ and $2i$, or it contains $2i$ but not $2i - 1$, or it contains both $2i - 1$ and $2i$. Furthermore, if we know for each $i \in [n/2]$ which of these three options it satisfies, then we know the whole subset $S$.

Thus, the following simple algorithm constructs every shadowed subset of $[n]$: For each $i \in [n/2]$, we decide whether our subset should contain none of the two integers $2i - 1$ and $2i$, or it should contain $2i$ but not $2i - 1$, or it should contain both $2i - 1$ and $2i$. There are clearly 3 options to choose from in this decision. Thus, in total, there are $3^{n/2}$ possible shadowed subsets of $[n]$ (because we are making this decision once for each of the $n/2$ elements $i$ of $[n/2]$). This completes our informal proof.

This argument can be translated into a formal proof (by bijection) in the same way as this was done in our solution to Exercise 4(a) above. Let me be very brief: Let $\mathcal{A}$ be the set of all shadowed subsets of $[n]$. We must show that $|\mathcal{A}| = 3^{n/2}$. It will suffice to exhibit a bijection $\mathcal{A} \rightarrow [3]^{[n/2]}$.

We define such a bijection $\Xi : \mathcal{A} \rightarrow [3]^{[n/2]}$ as follows: It should send any shadowed subset $S$ of $[n]$ to the map $f : [n/2] \rightarrow [3]$ that sends each $i \in [n/2]$ to

$$
\begin{cases}
1, & \text{if } S \text{ contains none of } 2i - 1 \text{ and } 2i; \\
2, & \text{if } S \text{ contains } 2i \text{ but not } 2i - 1; \\
3, & \text{if } S \text{ contains both } 2i - 1 \text{ and } 2i.
\end{cases}
$$

The reader can easily check that this $\Xi$ is well-defined and has an inverse, and that completes the proof.

(b) The number of shadowed subsets of $[n]$ is $3^{(n-1)/2}$.

Proof. We know that $n \neq 0$ (since $n$ is odd); thus, $n$ is a positive integer (since $n \in \mathbb{N}$). Hence, $n - 1 \in \mathbb{N}$. Moreover, $n - 1$ is even (since $n$ is odd). Hence, Exercise 4(a) (applied to $n - 1$ instead of $n$) shows that the number of shadowed subsets of $[n - 1]$ is $3^{(n-1)/2}$.

But any shadowed subset of $[n]$ must be a subset of $[n - 1]$. Hence, the shadowed subsets of $[n]$ are precisely the shadowed subsets of $[n - 1]$; consequently,

$^4$Proof. Let $S$ be a shadowed subset of $[n]$. We must show that $S$ is a subset of $[n - 1]$.

We know that $S$ is shadowed. In other words, whenever an odd integer $i$ belongs to $S$, the next integer $i + 1$ must also belong to $S$. Applying this to $i = n$, we conclude that if $n$ belongs to $S$, then $n + 1$ must also belong to $S$ (since $n$ is an odd integer). Therefore, $n$ cannot belong to $S$ (since $n + 1$ cannot belong to $S$ (because $S$ is a subset of $[n]$, and $n + 1$ does not belong to $[n]$)). Therefore, $S$ is a subset of $[n] \setminus \{n\} = [n - 1]$. Qed.
their number is $3^{(n-1)/2}$ (because we have just shown that the number of shadowed subsets of $[n-1]$ is $3^{(n-1)/2}$). This completes the proof. \qed

0.5. Counting smords (Smirnov words, or Carlitz words)

Exercise 5. Let $n$ and $k$ be positive integers. A $k$-smord will mean a $k$-tuple $(a_1,a_2,\ldots,a_k) \in [n]^k$ such that no two consecutive entries of the $k$-tuple are equal (i.e., we have $a_i \neq a_{i+1}$ for all $i \in [k-1]$). For example, $(3,1,3,2)$ is a 4-smord (when $n \geq 3$), but $(1,3,3,2)$ is not.

(a) Compute the number of all $k$-smords.
(b) A $k$-smord $(a_1,a_2,\ldots,a_k)$ is said to be rounded if it furthermore satisfies $a_k \neq a_1$. Compute the number of all rounded $k$-smords.

Before I come to the solution of this exercise, let me quickly comment on where it comes from. What I call “$k$-smords” in Exercise 5 is usually called “Smirnov words” or “Carlitz words” (of length $k$, over the alphabet $[n]$). Generally, combinatorialists often use the word “word of length $k$ over an alphabet $A$” as a synonym for “$k$-tuple of elements of $A$”, with no linguistic or semantic connotations in mind.

The exercise, however, has a deeper significance in combinatorics: It provides two simple examples for the computation of a chromatic polynomial. I hope we will come to see the general case in class.

TODO: Say more about chromatic polynomials!

Solution to Exercise 5 (sketched). (a) The number of all $k$-smords is $n(n-1)^{k-1}$.

Proof. A $k$-smord is simply a $k$-tuple of elements of $[n]$ such that each entry (apart from the first) is distinct from the previous entry. Thus, the following algorithm constructs each $k$-smord:

- First, choose the first entry of the $k$-smord. There are $n$ choices here.
- Then, choose the second entry of the $k$-smord. There are $n-1$ choices for this, because it has to be distinct from the previous entry.
- Then, choose the third entry of the $k$-smord. There are $n-1$ choices for this, because it has to be distinct from the previous entry.
- And so on, until all entries have been chosen.

Thus, in total, there are

$$n(n-1)(n-1)\cdots(n-1) = n(n-1)^{k-1}$$

k-1 times

\footnote{I have abbreviated this to “smords” in the exercise to make it harder to google. The definition of “smord” in Urban Dictionary is an (unintended) red herring.}


ways to perform this algorithm. Hence, the number of all \( k \)-smords is \( n (n - 1)^{k - 1} \).

(b) The number of all rounded \( k \)-smords is \( (n - 1)^k + (-1)^k (n - 1) \).

Proof. Let \( r_k(n) \) denote the number of all rounded \( k \)-smords. We must prove that

\[
  r_k(n) = (n - 1)^k + (-1)^k (n - 1). \tag{5}
\]

Let us forget that we fixed \( k \). We shall now prove (5) by induction over \( k \):

Induction base: A 1-smord (a) is rounded if and only if it satisfies \( a \neq a \) (by the definition of “rounded”); thus, there exist no rounded 1-smords (because \( a \neq a \) never holds). Hence, the number of all rounded 1-smords is 0. In other words, \( r_1(n) = 0 \) (since \( r_1(n) \) was defined to be the number of all rounded 1-smords). Comparing this with \( (n - 1)^1 + (-1)^1 (n - 1) = 0 \), we obtain \( r_1(n) = (n - 1)^1 + (-1)^1 (n - 1) \). In other words, (5) holds for \( k = 1 \). This completes the induction base.

Induction step: You have seen lots of induction steps by now, so let me take away one piece of railing for the sake of brevity. Namely, instead of stepping “from \( k = m \) to \( k = m + 1 \)”, I shall simply “step from \( k \) to \( k + 1 \)”. This is just a matter of notation, which at this point should not be too confusing any longer.

So let \( k \) be a positive integer, and assume (as our induction hypothesis) that (5) holds “for this particular \( k \)” (that is, we have \( r_k(n) = (n - 1)^k + (-1)^k (n - 1) \)). Then, we must show that (5) holds “for \( k + 1 \) as well” (that is, we must show that \( r_{k+1}(n) = (n - 1)^{k+1} + (-1)^{k+1} (n - 1) \)).

We say that a \((k + 1)\)-smord is non-rounded if it is not rounded. (Duh.)

Exercise 5(a) (applied to \( k + 1 \) instead of \( k \)) shows that the number of all \((k + 1)\)-smords is \( n (n - 1)^{(k+1)-1} = n (n - 1)^k \). Hence,

\[
  n (n - 1)^k = (\text{the number of all} (k + 1) - \text{smords})
  = (\text{the number of all rounded} (k + 1) - \text{smords})
  + (\text{the number of all non-rounded} (k + 1) - \text{smords}). \tag{6}
\]

We shall now find the number of all non-rounded \((k + 1)\)-smords.

A \((k + 1)\)-smord \((a_1, a_2, \ldots, a_{k+1})\) is rounded if and only if it satisfies \( a_{k+1} \neq a_1 \) (by the definition of “rounded”). Hence, \((k + 1)\)-smord \((a_1, a_2, \ldots, a_{k+1})\) is non-rounded if and only if it satisfies \( a_{k+1} = a_1 \). Thus, a non-rounded \((k + 1)\)-smord \((a_1, a_2, \ldots, a_{k+1})\) is uniquely determined by its first \( k \) entries \( a_1, a_2, \ldots, a_k \). Moreover, these first \( k \) entries must themselves form a \( k \)-smord (since \( a_i \neq a_{i+1} \) holds for all \( i \) \( \in \) \([k]\) and therefore also for all \( i \) \( \in \) \([k-1]\), and this \( k \)-smord \((a_1, a_2, \ldots, a_k)\) is rounded (because \( a_i \neq a_{i+1} \) for all \( i \) \( \in \) \([k]\), whence \( a_k \neq a_{k+1} = a_1 \), but this says precisely that the \( k \)-smord \((a_1, a_2, \ldots, a_k)\) is rounded). Hence, we can define a map

\[
  \phi: \{\text{non-rounded} (k + 1) - \text{smords}\} \rightarrow \{\text{rounded} k - \text{smords}\},
  (a_1, a_2, \ldots, a_{k+1}) \mapsto (a_1, a_2, \ldots, a_k).
\]

Conversely, if \((a_1, a_2, \ldots, a_k)\) is a rounded \( k \)-smord, then \((a_1, a_2, \ldots, a_k, a_1)\) is a non-rounded \((k + 1)\)-smord (in fact, it is a \((k + 1)\)-smord because the roundedness of
\((a_1, a_2, \ldots, a_k)\) leads to \(a_k \neq a_1\); and it is non-rounded because \(a_1 = a_1\). Thus, we can define a map

\[
\psi : \{\text{rounded } k\text{-smords}\} \to \{\text{non-rounded } (k + 1)\text{-smords}\},
\]

\[
(a_1, a_2, \ldots, a_k) \mapsto (a_1, a_2, \ldots, a_k, a_1).
\]

The two maps \(\phi\) and \(\psi\) are mutually inverse (to check this, just remember that any non-rounded \((k + 1)\)-smord \((a_1, a_2, \ldots, a_{k+1})\) must satisfy \(a_{k+1} = a_1\), so it is identical with \((a_1, a_2, \ldots, a_k, a_1)\), and thus are bijections. Hence, we have found a bijection from \{non-rounded \((k + 1)\)-smords\} to \{rounded \(k\)-smords\} (namely, \(\phi\)). Therefore,

\[
\text{(the number of all non-rounded } (k + 1)\text{-smords)} = \text{(the number of all rounded } k\text{-smords)}.
\]

Thus, (6) becomes

\[
n (n - 1)^k = \underbrace{r_{k+1}(n)} + \underbrace{r_k(n)} = \underbrace{r_{k+1}(n)} + \underbrace{r_k(n)} = n (n - 1)^k - \underbrace{(n - 1)^k + (-1)^k (n - 1)}.
\]

Therefore,

\[
\begin{align*}
  r_{k+1}(n) &= n (n - 1)^k - \frac{r_k(n)}{=(n-1)^k + (-1)^k (n-1)} (n - 1) \\
  &= \frac{n (n - 1)^k - (n - 1)^k - (-1)^k (n - 1)}{=(n-1)(n-1)^k + (-1)^k (n-1) = (-1)^{k+1}} \\
  &= (n - 1)^{k+1} + (-1)^{k+1} (n - 1).
\end{align*}
\]

In other words, (5) holds “for \(k + 1\) as well”. This completes the induction step. Thus, the induction proof of (5) is finished, and with it the solution of Exercise 5 (b).

0.6. Necklaces 2: rotational equivalence of tuples

Let us recall a basic property of maps (proven in Exercise 6 (a) on Math 4990 homework set #2): If \(S\) is a set, and if \(f : S \to S\) a map, then

\[
f^n \circ f^m = f^{n+m}
\]

for each \(n, m \in \mathbb{N}\).
Exercise 6. This continues Exercise 7 from Math 4990 homework set #2.
Let \( n \) be a positive integer. Let \( X \) be a set.
We define a map \( c : X^n \to X^n \) by
\[
c(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1)
\]
for all \( (x_1, x_2, \ldots, x_n) \in X^n \).
(In other words, the map \( c \) transforms any \( n \)-tuple \( (x_1, x_2, \ldots, x_n) \in X^n \) by “rotating” it one step to the left, or, equivalently, moving its first entry to the last position.)

For two \( n \)-tuples \( x \) and \( y \), we say that \( x \sim y \) if there exists some \( k \in \mathbb{N} \) such that \( y = c^k(x) \). (For example, \((1, 5, 2, 4) \sim (2, 4, 1, 5) \), because \((2, 4, 1, 5) = c^2(1, 5, 2, 4) \).)

(a) Prove that \( \sim \) is an equivalence relation, i.e., is reflexive, transitive and symmetric. (For example, symmetry boils down to showing that if there exists some \( k \in \mathbb{N} \) satisfying \( y = c^k(x) \), then there exists some \( \ell \in \mathbb{N} \) satisfying \( x = c^\ell(y) \).)

(b) An \( n \)-necklace (over \( X \)) shall mean a \( \sim \)-equivalence class. We denote the \( \sim \)-equivalence class of a tuple \( x \in X^n \) by \( [x]_\sim \).
Let \( x \in X^n \) be an \( n \)-tuple. Let \( m \) be the smallest nonzero period of the \( n \)-tuple \( x \in X^n \).
Prove that \( [x]_\sim = \{c^0(x), c^1(x), \ldots, c^{m-1}(x)\} \).
(c) Show that the \( m \) tuples \( c^0(x), c^1(x), \ldots, c^{m-1}(x) \) are distinct. Conclude that \( |[x]_\sim| = m \).

Solution to Exercise 6. Before we properly start solving this exercise, let us make some basic observations:

Observation 1: We have \( c^n(x) = x \) for each \( x \in X^n \).

[Proof of Observation 1: Let \( x \in X^n \). We have proven \( c^n(x) = x \) during our solution to Exercise 7 (d) on Math 4990 homework set #2. Thus, Observation 1 follows.]

Observation 2: Let \( x \in X^n \). Let \( p \in \mathbb{N} \) be such that \( c^p(x) = x \). Then, \( c^{kp}(x) = x \) for each \( k \in \mathbb{N} \).

[Proof of Observation 2: Observation 2 is intuitively obvious: All it says is that if applying the map \( c \) to \( x \) a total of \( p \) times brings you back to \( x \), then applying the map \( c \) to \( x \) a total of \( kp \) times brings you back to \( x \) as well. This intuition can easily be translated into a rigorous argument:
We shall prove Observation 2 by induction over \( k \):
Induction base: We have \( c^{0p} = c^0 = \text{id}_{X^n} \), so that \( c^{0p}(x) = \text{id}_{X^n}(x) = x \). Thus, Observation 2 holds for \( k = 0 \). This completes the induction base.
Induction step: Let \( m \in \mathbb{N} \). Assume that Observation 2 holds for \( k = m \). We must prove that Observation 2 holds for \( k = m + 1 \).]
Let \( x \in X^n \). Let \( p \in \mathbb{N} \) be such that \( c^p (x) = x \). Then, \( c^{mp} (x) = x \) (since Observation 2 holds for \( k = m \)). But \( n \) \( (a) \) \( \text{reflexive} \)

Observation 2 holds for \( k = f \) and \( n \) \( (b) \) \( \text{symmetric} \) and \( \text{transitive} \). In other words, the relation \( \sim \) is an equivalence relation. This solves Exercise \( 6(a) \).

\[ c^{(m+1)p} (x) = (c^{mp} \circ c^p) (x) = c^{mp} \left( c^p (x) \right) = c^{mp} (x) = x. \]

In other words, Observation 2 holds for \( k = m + 1 \). This completes the induction step. Thus, Observation 2 is proven.

Now, we must show that \( \sim \) is an equivalence relation. Indeed, the relation \( \sim \) is reflexive, symmetric and transitive. In other words, the relation \( \sim \) is an equivalence relation. This solves Exercise \( 6(a) \).

---

\[ \text{Proof. Let } x \in X^n. \text{ We shall show that } x \sim x. \]

Indeed, \( c^{0} = \text{id}_{X^n} \), so that \( c^{0} (x) = \text{id}_{X^n} (x) = x \). Hence, there exists some \( k \in \mathbb{N} \) such that \( x = c^k (x) \) (namely, \( k = 0 \)). In other words, \( x \sim x \) (by the definition of the relation \( \sim \)).

Now, forget that we fixed \( x \). We thus have shown that every \( x \in X^n \) satisfies \( x \sim x \). In other words, the relation \( \sim \) is reflexive.

\[ \text{Proof. Let } x \in X^n \) and \( y \in X^n \) be such that \( x \sim y \). We shall show that \( y \sim x \).

Indeed, we have \( x \sim y \). In other words, there exists some \( k \in \mathbb{N} \) such that \( y = c^k (x) \) (by the definition of the relation \( \sim \)). Consider such a \( k \), and denote it by \( u \). Thus, \( u \in \mathbb{N} \) satisfies \( y = c^u (x) \).

Observation 1 yields \( c^u (x) = x \). Hence, Observation 2 (applied to \( p = n \) and \( k = u \)) yields \( c^{un} (x) = x \). But \( n \) is positive; hence, \( n \geq 1 \) and thus \( un \geq u1 = u \). Hence, \( un - u \in \mathbb{N} \). Applying \( (b) \) to \( X^n \), \( c \), \( un - u \) and \( u \) instead of \( S \), \( f \), \( n \) and \( m \), we obtain \( c^{un-u} \circ c^u = c^{(un-u)+u} = c^{un} \). Thus,

\[ (c^{un-u} \circ c^u) (x) = c^{un} (x) \]

\[ x = (c^{un} \circ c^u) (x) = c^{un-u} \left( \frac{c^u (x)}{\frac{y}{\text{by the definition of the relation } \sim}} \right) = c^{un-u} (y) \]

there exists some \( k \in \mathbb{N} \) such that \( x = c^k (y) \) (namely, \( k = un - u \)). In other words, \( y \sim x \) (by the definition of the relation \( \sim \)).

Now, forget that we fixed \( x \) and \( y \). We thus have shown that if \( x \in X^n \) and \( y \in X^n \) satisfy \( x \sim y \), then \( y \sim x \). In other words, the relation \( \sim \) is symmetric.

---

\[ \text{Proof. Let } x \in X^n \), \( y \in X^n \) and \( z \in X^n \) be such that \( x \sim y \) and \( y \sim z \). We shall show that \( x \sim z \).

Indeed, we have \( x \sim y \). In other words, there exists some \( k \in \mathbb{N} \) such that \( y = c^k (x) \) (by the definition of the relation \( \sim \)). Consider such a \( k \), and denote it by \( v \). Thus, \( u \in \mathbb{N} \) satisfies \( y = c^v (x) \).

Also, we have \( y \sim z \). In other words, there exists some \( k \in \mathbb{N} \) such that \( z = c^k (y) \) (by the definition of the relation \( \sim \)). Consider such a \( k \), and denote it by \( v \). Thus, \( v \in \mathbb{N} \) satisfies \( z = c^v (y) \).

Applying \( (b) \) to \( X^n \), \( c \), \( v \) and \( u \) instead of \( S \), \( f \), \( n \) and \( m \), we obtain \( c^v \circ c^u = c^{v+u} \). Thus,

\[ (c^v \circ c^u) (x) = c^{v+u} (x) \]

In view of \( (c^v \circ c^u) (x) = c^v \left( \frac{c^u (x)}{\frac{y}{\text{by the definition of the relation } \sim}} \right) = c^v (y) = z \), this rewrites as \( z = c^{v+u} (x) \). Thus, there exists some \( k \in \mathbb{N} \) such that \( z = c^k (x) \) (namely, \( k = v + u \)). In other words, \( x \sim z \) (by the definition of the relation \( \sim \)).

Now, forget that we fixed \( x \), \( y \) and \( z \). We thus have shown that if \( x \in X^n \), \( y \in X^n \) and \( z \in X^n \) satisfy \( x \sim y \) and \( y \sim z \), then \( x \sim z \). In other words, the relation \( \sim \) is transitive.
(b) The number \( m \) is the smallest nonzero period of the \( n \)-tuple \( x \in X^n \). In particular, \( m \) is a period of \( x \). In other words, \( m \in \mathbb{N} \) and \( c^m(x) = x \).

The definition of the equivalence class \([x]_\sim \) of \( x \) shows that
\[
[x]_\sim = \{ y \in X^n \mid y \sim x \}.
\]

Let \( S \) denote the set \( \{ c^0(x), c^1(x), \ldots, c^{m-1}(x) \} \). Then, \( S \subseteq [x]_\sim \).

Observation 3: We have \( c^k(x) \in S \) for each \( k \in \mathbb{N} \).

Proof of Observation 3: We proceed by strong induction over \( k \):

Induction step: Let \( h \in \mathbb{N} \). Assume that Observation 3 holds whenever \( k < h \). We now must prove that Observation 3 holds for \( k = h \). In other words, we must prove that \( c^h(x) \in S \).

If \( h < m \), then this is obvious. Hence, for the rest of this proof (i.e., of the induction step), we WLOG assume that we don’t have \( h < m \). Thus, \( h \geq m \), so that \( h - m \in \mathbb{N} \).

We know that \( m \) is nonzero, and therefore positive (since \( m \in \mathbb{N} \)). Hence, \( h - m < h \). Therefore (and because of \( h - m \in \mathbb{N} \)), we can apply Observation 3 to \( k = h - m \) (since we have assumed that Observation 3 holds whenever \( k < h \)). We thus obtain \( c^{h-m}(x) \in S \).

But \( c^0 \) (applied to \( X^n \), \( c \), \( h - m \) and \( m \) instead of \( S \), \( f \), \( n \) and \( m \)) shows that \( c^{h-m} \circ c^m = c^{(h-m)+m} = c^h \). Hence, \( (c^{h-m} \circ c^m)(x) = c^h(x) \), so that
\[
c^h(x) = c^{h-m} \circ c^m(x) = c^{h-m} \left( c^m(x) \right) = c^{h-m} = (x) \in S.
\]

In other words, Observation 3 holds for \( k = h \). This completes the induction step. Observation 3 is thus proven.

Now, it is easy to see that \([x]_\sim \subseteq S \). Combining this with \( S \subseteq [x]_\sim \), we obtain \([x]_\sim = S \) (by the definition of \( S \)). This solves Exercise 6 (b).

---

9Proof. Let \( s \in S \). Then, \( s \in S = \{ c^0(x), c^1(x), \ldots, c^{m-1}(x) \} \). In other words, \( s = c^i(x) \) for some \( i \in \{0,1,\ldots,m-1\} \). Consider this \( i \).

Hence, \( s \in X^n \). Furthermore, there exists some \( k \in \mathbb{N} \) such that \( s = c^k(x) \) (namely, \( k = i \)). In other words, \( x \sim s \) (by the definition of the relation \( \sim \)). Hence, \( s \sim x \) (since the relation \( \sim \) is symmetric). Therefore, \( s \in \{ y \in X^n \mid y \sim x \} \). In light of (8), this rewrites as \( s \in [x]_\sim \).

Now, forget that we fixed \( s \). We thus have shown that \( s \in [x]_\sim \) for each \( s \in S \). In other words, \( S \subseteq [x]_\sim \).

10Proof. Assume that \( h < m \). Thus, \( h \in \{0,1,\ldots,m-1\} \) (since \( h \in \mathbb{N} \)), and thus \( c^h(x) \in \{ c^0(x), c^1(x), \ldots, c^{m-1}(x) \} = S \). qed.

11Proof. Let \( s \in [x]_\sim \). Then, \( s \in [x]_\sim = \{ y \in X^n \mid y \sim x \} \) (by (8)). In other words, \( s \in X^n \) and \( s \sim x \). From \( s \sim x \), we obtain \( x \sim s \) (since the relation \( \sim \) is symmetric). In other words, there exists some \( k \in \mathbb{N} \) such that \( s = c^k(x) \) (by the definition of the relation \( \sim \)). Consider this \( k \).

Now, Observation 3 yields \( c^k(x) \in S \). Hence, \( s = c^k(x) \in S \).

Now, forget that we fixed \( s \). We thus have shown that \( s \in S \) for each \( s \in [x]_\sim \). In other words, \([x]_\sim \subseteq S \).
(c) We observe that \( m \in \mathbb{N} \) and \( c^m(x) = x \) (as we have already seen in the solution to part (b)).

Now, we are going to show the following:

\[ \text{Observation 4: Let } i \text{ and } j \text{ be two distinct elements of } \{0, 1, \ldots, m - 1\}. \]

Then, \( c^i(x) \neq c^j(x) \).

[Proof of Observation 4: We WLOG assume that \( i \leq j \) (since otherwise, we can simply switch \( i \) with \( j \) to ensure that \( i \leq j \)). Hence, \( i < j \) (since \( i \) and \( j \) are distinct).

Assume (for the sake of contradiction) that \( c^i(x) = c^j(x) \). Since \( j \in \{0, 1, \ldots, m - 1\} \), we have \( j \leq m - 1 < m \). Hence, \( \left( m - \frac{j}{m} \right) + i > (m - m) + i = i \geq 0. \)

Also \( m - j \in \mathbb{N} \) (since \( j < m \)). Therefore, (7) (applied to \( X^n, c, m - j \) and \( j \) instead of \( S, f, n \) and \( m \)) shows that \( c^{m-j} \circ c^i = c^{(m-j)+j} = c^m \). Hence, \( (c^{m-j} \circ c^i)(x) = c^m(x) = x \). Therefore,

\[
x = \left( c^{m-j} \circ c^i \right)(x) = c^{m-j} \left( c^i(x) \right).
\]

On the other hand, (7) (applied to \( X^n, c, m - j \) and \( i \) instead of \( S, f, n \) and \( m \)) shows that \( c^{m-j} \circ c^i = c^{(m-j)+i} \). Hence, \( (c^{m-j} \circ c^i)(x) = c^{(m-j)+i}(x) \). Therefore,

\[
c^{(m-j)+i}(x) = \left( c^{m-j} \circ c^i \right)(x) = c^{m-j} \left( c^i(x) \right) = c^{m-j} \left( \underbrace{c^i(x)}_{=c^i(x)} \right) = c^{m-j} \left( c^i(x) \right) = x
\]

(by (7)).

Now, the integer \( (m - j) + i \) belongs to \( \mathbb{N} \) (since \( (m - j) + i > 0 \)) and satisfies \( c^{(m-j)+i}(x) = x \). In other words, \( (m - j) + i \) is a period of \( x \) (by the definition of a “period”). Moreover, this period \( (m - j) + i \) is nonzero (since \( (m - j) + i > 0 \)).

Recall that \( m \) is the smallest nonzero period of the \( n \)-tuple \( x \in X^n \). Hence, every nonzero period \( p \) of \( x \) satisfies \( p \geq m \). Applying this to \( p = (m - j) + i \), we obtain \( (m - j) + i \geq m \) (since \( (m - j) + i \) is a nonzero period of \( x \)). This contradicts \( (m - j) + i < (m - j) + j = m \). This contradiction shows that our assumption \( c^i(x) = c^j(x) \) was wrong. Hence, \( c^i(x) \neq c^j(x) \). This proves Observation 4.]

Observation 4 shows that the \( m \) tuples \( c^0(x), c^1(x), \ldots, c^{m-1}(x) \) are distinct. Hence, the set \( \{c^0(x), c^1(x), \ldots, c^{m-1}(x)\} \) contains \( m \) distinct elements. Therefore, \( \left| \{c^0(x), c^1(x), \ldots, c^{m-1}(x)\} \right| = m. \) But Exercise \( 6(b) \) shows that \( [x]_\sim = \{c^0(x), c^1(x), \ldots, c^{m-1}(x)\} \). Thus, \( \left| [x]_\sim \right| = \left| \{c^0(x), c^1(x), \ldots, c^{m-1}(x)\} \right| = m. \) This solves Exercise \( 6(c) \).