Math 4990 Fall 2017 (Darij Grinberg): homework set 7

Due date: Tuesday 14 Nov 2017 at the beginning of class, or before that by email or moodle

Please solve at most 4 of the 7 exercises!

0.1. A generalized principle of inclusion/exclusion

**Exercise 1.** Let \( n \in \mathbb{N} \). Let \( S \) be a finite set. Let \( A_1, A_2, \ldots, A_n \) be finite subsets of \( S \). Let \( k \in \mathbb{N} \). Let \( S_k \) be the set of all elements of \( S \) that belong to exactly \( k \) of the subsets \( A_1, A_2, \ldots, A_n \). (In other words, let \( S_k = \{ s \in S \mid \text{the number of } i \in [n] \text{ satisfying } s \in A_i \text{ equals } k \} \).) Prove that

\[
|S_k| = \sum_{I \subseteq [n]} (-1)^{|I|-k} \binom{|I|}{k} \left| \bigcap_{i \in I} A_i \right|.
\]

Note that the principle of inclusion and exclusion (see, e.g., [Galvin17, §16]) is the particular case of Exercise 1 for \( k = 0 \) (since \( S_0 = S \setminus \bigcup_{i=1}^n A_i \)).

0.2. Summing fixed point numbers of permutations

Recall that for any \( n \in \mathbb{N} \), we let \( S_n \) denote the set of all permutations of \([n]\).

If \( S \) is a finite set, and if \( f : S \rightarrow S \) is a map, then we let \( \text{Fix } f \) denote the set of all fixed points of \( f \). (That is, \( \text{Fix } f = \{ s \in S \mid f(s) = s \} \).)

**Exercise 2.** Let \( n \) be a positive integer. Prove that \( \sum_{w \in S_n} |\text{Fix } w| = n! \).

**[Hint: Rewrite }|\text{Fix } w| \text{ as } \sum_{i \in [n]} [w(i) = i].**

(In other words, this exercise states that the average number of fixed points of a permutation of \([n]\) is 1.)

0.3. Transpositions \( t_{i,j} \) generate permutations

Recall a basic notation regarding permutations:

**Definition 0.1.** Let \( n \in \mathbb{N} \). Let \( i \) and \( j \) be two distinct elements of \([n]\). We let \( t_{i,j} \) be the permutation in \( S_n \) which switches \( i \) with \( j \) while leaving all other elements of \([n]\) unchanged. Such a permutation is called a transposition.

**Exercise 3.** Let \( n \in \mathbb{N} \). Prove that each permutation in \( S_n \) can be written as a composition of some of the transpositions \( t_{1,2}, t_{1,3}, \ldots, t_{1,n} \).

(Note that this composition can be empty – in which case it is understood to be \( \text{id} \) –, and it can contain any given transposition multiple times.)
0.4. V-permutations as products of cycles

Recall the following notation:

**Definition 0.2.** Let $X$ be a set. Let $k$ be a positive integer. Let $i_1, i_2, \ldots, i_k$ be $k$ distinct elements of $X$. We define $\text{cyc}_{i_1, i_2, \ldots, i_k}$ to be the permutation of $X$ that sends $i_1, i_2, \ldots, i_k$ to $i_2, i_3, \ldots, i_k, i_1$, respectively, while leaving all other elements of $X$ fixed. In other words, we define $\text{cyc}_{i_1, i_2, \ldots, i_k}$ to be the permutation of $X$ given by

$$\text{cyc}_{i_1, i_2, \ldots, i_k}(p) = \begin{cases} i_{j+1}, & \text{if } p = i_j \text{ for some } j \in \{1, 2, \ldots, k\}; \\ p, & \text{otherwise} \end{cases}$$

for every $p \in X$,

where $i_{k+1}$ means $i_1$.

**Exercise 4.** Let $n \in \mathbb{N}$. For each $r \in [n]$, let $c_r$ denote the permutation $\text{cyc}_{r, r-1, 2, 1} \in S_n$. (Thus, $c_1 = \text{cyc}_1 = \text{id}$ and $c_2 = \text{cyc}_{2, 1} = s_1$.)

Let $G = \{g_1 < g_2 < \ldots < g_p\}$ be a subset of $[n]$. Let $\sigma \in S_n$ be the permutation $c_{g_1} \circ c_{g_2} \circ \cdots \circ c_{g_p}$.

Prove the following:

(a) We have $\sigma(1) > \sigma(2) > \cdots > \sigma(p)$.

(b) We have $\sigma([p]) = G$.

(c) We have $\sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(n)$.

(Note that a chain of inequalities that involves less than two numbers is considered to be vacuously true. For example, Exercise 4(c) is vacuously true when $p = n - 1$ and also when $p = n$.)

Permutations $\sigma \in S_n$ satisfying the inequalities $\sigma(1) > \sigma(2) > \cdots > \sigma(p)$ and $\sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(n)$ for some $p \in \{0, 1, \ldots, n\}$ are known as “V-permutations” (as their plot looks somewhat like the letter “V”: first decreasing for a while, then increasing). Can you guess how permutations $\sigma \in S_n$ satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p + 1) > \sigma(p + 2) > \cdots > \sigma(n)$ are called?

0.5. Lexicographic comparison of permutations

**Definition 0.3.** Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a permutation. For any $i \in [n]$, we let $\ell_i(\sigma)$ denote the number of $j \in \{i + 1, i + 2, \ldots, n\}$ such that $\sigma(i) > \sigma(j)$.

For example, if $\sigma$ is the permutation of $[5]$ written in one-line notation as $[4, 1, 5, 2, 3]$, then $\ell_1(\sigma) = 3$, $\ell_2(\sigma) = 0$, $\ell_3(\sigma) = 2$, $\ell_4(\sigma) = 0$ and $\ell_5(\sigma) = 0$. 

Definition 0.4. Let \( n \in \mathbb{N} \). Let \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\) be two \( n \)-tuples of integers. We say that \((a_1, a_2, \ldots, a_n) <_{\text{lex}} (b_1, b_2, \ldots, b_n)\) if and only if there exists some \( k \in [n] \) such that \( a_k \neq b_k \), and the smallest such \( k \) satisfies \( a_k < b_k \).

For example, \((4, 1, 2, 5) <_{\text{lex}} (4, 1, 3, 0)\) and \((1, 1, 0, 1) <_{\text{lex}} (2, 0, 0, 0)\). The relation \(<_{\text{lex}}\) is usually pronounced “is lexicographically smaller than”; the word “lexigraphic” comes from the idea that if numbers were letters, then a “word” \( a_1 a_2 \cdots a_n \) would appear earlier in a dictionary than \( b_1 b_2 \cdots b_n \) if and only if \((a_1, a_2, \ldots, a_n) <_{\text{lex}} (b_1, b_2, \ldots, b_n)\).

Exercise 5. Let \( n \in \mathbb{N} \). Let \( \sigma \in S_n \) and \( \tau \in S_n \). Prove the following:

(a) If \((\sigma(1), \sigma(2), \ldots, \sigma(n)) <_{\text{lex}} (\tau(1), \tau(2), \ldots, \tau(n))\), then \((\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) <_{\text{lex}} (\ell_1(\tau), \ell_2(\tau), \ldots, \ell_n(\tau))\).

(b) If \((\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) = (\ell_1(\tau), \ell_2(\tau), \ldots, \ell_n(\tau))\), then \(\sigma = \tau\).

0.6. Comparing subsets of \([n]\)

If \( I \) and \( J \) are two finite sets of integers, then we write \( I \leq_{\#} J \) if and only if the following two properties hold:

- We have \(|I| \geq |J|\).
- For every \( r \in \{1, 2, \ldots, |J|\} \), the \( r \)-th smallest element of \( I \) is \( \leq \) to the \( r \)-th smallest element of \( J \).

For example, \( \{2, 4\} \leq_{\#} \{2, 5\} \) and \( \{1, 3\} \leq_{\#} \{2, 4\} \) and \( \{1, 3, 5\} \leq_{\#} \{2, 4\} \). (But not \( \{1, 3\} \leq_{\#} \{2, 4, 5\} \).)

Exercise 6. Let \( n \in \mathbb{N} \). Let \( I \) and \( J \) be two subsets of \([n]\).

(a) For every subset \( S \) of \([n]\) and every \( \ell \in [n] \), let \( a_S(\ell) \) denote the number of all elements of \( S \) that are \( \leq \ell \). Prove that \( I \leq_{\#} J \) holds if and only if every \( \ell \in [n] \) satisfies \( a_I(\ell) \geq a_J(\ell) \).

(b) Prove that \( I \leq_{\#} J \) if and only if \([n] \setminus J \leq_{\#} [n] \setminus I \).

Remark 0.5. Recall that we have defined a Dyck word as a list \( w \) of \( 2n \) numbers, exactly \( n \) of which are 0’s while the other \( n \) are 1’s, and having the property that for each \( k \in [2n] \), the number of 0’s among the first \( k \) entries of \( w \) is \( \leq \) to the number of 1’s among the first \( k \) entries of \( w \).

It is not hard to see the connection between the relation \( \leq_{\#} \) and Dyck words: Let \( w = (w_1, w_2, \ldots, w_{2n}) \in \{0, 1\}^{2n} \) be a list of \( 2n \) numbers, exactly \( n \) of which are 0’s while the other \( n \) are 1’s. Then, \( w \) is a Dyck word if and only if

\[
\{ i \in [2n] \mid w_i = 1 \} \leq_{\#} \{ i \in [2n] \mid w_i = 0 \}
\]
(in other words, for every \( r \in [n] \), the \( r \)-th appearance of 1 in \( w \) precedes the \( r \)-th appearance of 0 in \( w \)).

0.7. A rigorous approach to the existence of a cycle decomposition

The purpose of the following exercise is to give a rigorous proof of the fact that any permutation can be decomposed into disjoint cycles.

**Exercise 7.** Let \( X \) be a finite set. Let \( \sigma \) be a permutation of \( X \).

Define a binary relation \( \sim \) on the set \( X \) as follows: For two elements \( x, y \in X \), we set \( x \sim y \) if and only if there exists some \( k \in \mathbb{N} \) such that \( y = \sigma^k(x) \).

(a) Prove that \( \sim \) is an equivalence relation.

For any \( x \in X \), we let \([x]_\sim\) denote the \( \sim \)-equivalence class of \( x \).

(b) For any \( x \in X \), prove that \([x]_\sim = \{\sigma^0(x), \sigma^1(x), \ldots, \sigma^{k-1}(x)\}\), where \( k = |[x]_\sim| \).

(c) For any \( \sim \)-equivalence class \( E \), let us define \( c_E \) to be the map

\[
X \to X, \quad x \mapsto \begin{cases} 
\sigma(x), & \text{if } x \in E; \\
x, & \text{if } x \notin E.
\end{cases}
\]

Prove that \( c_E \) is a permutation of \( X \).

(d) Prove that if \( E = [x]_\sim \) for some \( x \in X \), then \( c_E \) can be written as \( \text{cyc}_{\sigma^0(x), \sigma^1(x), \ldots, \sigma^{k-1}(x)} \), where \( k = |[x]_\sim| \). (Don’t forget to show that \( \sigma^0(x), \sigma^1(x), \ldots, \sigma^{k-1}(x) \) are distinct, so that \( \text{cyc}_{\sigma^0(x), \sigma^1(x), \ldots, \sigma^{k-1}(x)} \) is well-defined.)

(e) Let \( E_1, E_2, \ldots, E_m \) be all \( \sim \)-equivalence classes (listed without repetitions – that is, \( E_i \neq E_j \) whenever \( i \neq j \)). Prove that

\[
\sigma = c_{E_1} \circ c_{E_2} \circ \cdots \circ c_{E_m}.
\]

References


The numbering of theorems and formulas in this link might shift
when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see https://github.com/darijgr/detnotes/releases/tag/2019-01-10.