0.1. Strange integers

Exercise 1. For any \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \), define a rational number \( T(m, n) \) by

\[
T(m, n) = \frac{(2m)! (2n)!}{m! n! (m+n)!}.
\]

(a) Prove that \( 4T(m, n) = T(m+1, n) + T(m, n+1) \) for every \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \).

(b) Prove that \( T(m, n) \in \mathbb{N} \) for every \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \).

(c) Prove that \( T(m, n) \) is an even integer for every \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \) unless \( (m, n) = (0, 0) \).

(d) If \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \) are such that \( m+n \) is odd and \( m+n > 1 \), then prove that \( 4 \mid T(m, n) \).

[Hint: Don’t be afraid to use induction. Part (b) suggests that the numbers \( T(m, n) \) count something, but no one has so far discovered what; combinatorial proofs aren’t always the easiest to find. For (c), start by showing that \( \binom{2g}{g} \) is even whenever \( g \) is a positive integer. For (d), start by showing that \( \binom{2g-1}{g-1} \) is even whenever \( g > 1 \) is odd.]

Exercise 2. Let \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \). Let \( p = \min \{ m, n \} \).

(a) Prove that

\[
\sum_{k=-p}^{p} (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} = \binom{m+n}{m}.
\]

(b) Prove that

\[
T(m, n) = \sum_{k=-p}^{p} (-1)^k \binom{2m}{m+k} \binom{2n}{n-k},
\]

where \( T(m, n) \) is defined as in Exercise 1.

[Hint: Part (a) should follow from something done in class. Then, compare part (b) with part (a).]
0.2. The length of a permutation

**Definition 0.1.** Let $n \in \mathbb{N}$.

(a) We let $S_n$ denote the set of all permutations of $[n]$.

(b) Let $\sigma \in S_n$ be a permutation of $[n]$.

(c) An inversion of $\sigma$ means a pair $(i,j)$ of elements of $[n]$ satisfying $i < j$ and $\sigma(i) > \sigma(j)$.

(d) The length of $\sigma$ is defined to be the number of inversions of $\sigma$. This length is denoted by $\ell(\sigma)$.

(e) The sign of $\sigma$ is defined to be the integer $(-1)^{\ell(\sigma)}$. It is denoted by $(-1)^\sigma$.

**Exercise 3.** Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Let $n = pq$. Consider the permutation $\sigma \in S_n$ that maps $(i-1)q+j$ to $(j-1)p+i$ for every $i \in [p]$ and $j \in [q]$.

This permutation $\sigma$ can be visualized as follows: Fill in a $p \times q$-matrix $A$ with the entries $1, 2, \ldots, n$ by going row by row from top to bottom:

$$
A = \begin{pmatrix}
1 & 2 & 3 & \cdots & q \\
q+1 & q+2 & q+3 & \cdots & 2q \\
2q+1 & 2q+2 & 2q+3 & \cdots & 3q \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(p-1)q+1 & (p-1)q+2 & (p-1)q+3 & \cdots & pq
\end{pmatrix}.
$$

Fill in a $p \times q$-matrix $B$ with the entries $1, 2, \ldots, n$ by going column by column from left to right:

$$
B = \begin{pmatrix}
1 & p+1 & 2p+1 & \cdots & (q-1)p+1 \\
2 & p+2 & 2p+2 & \cdots & (q-1)p+2 \\
3 & p+3 & 2p+3 & \cdots & (q-1)p+3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p & 2p & 3p & \cdots & qp
\end{pmatrix}.
$$

The permutation $\sigma$ then sends each entry of $A$ to the corresponding entry of $B$.

Find the length $\ell(\sigma)$ of the permutation $\sigma$.

0.3. Two equal counts

**Exercise 4.** Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Prove that

$$
\text{(the number of all} \ (i,j) \in [n] \times [n] \ \text{such that} \ i \geq j > \sigma(i)) = \text{(the number of all} \ (i,j) \in [n] \times [n] \ \text{such that} \ \sigma(i) \geq j > i).
$$

0.4. Lehmer codes

Recall the following definition from the preceding homework set:
Definition 0.2. Let \( n \in \mathbb{N} \). Let \( \sigma \in S_n \) be a permutation. For any \( i \in [n] \), we let \( \ell_i(\sigma) \) denote the number of \( j \in \{i+1, i+2, \ldots, n\} \) such that \( \sigma(i) > \sigma(j) \).

Exercise 5. Let \( n \in \mathbb{N} \). Let \( G \) be the set of all \( n \)-tuples \((j_1, j_2, \ldots, j_n)\) of integers satisfying \( 0 \leq j_k \leq n-k \) for each \( k \in [n] \). (In other words, \( G = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-2\} \times \cdots \times \{0, 1, \ldots, n-n\} \).)

(a) For any \( \sigma \in S_n \) and \( i \in [n] \), prove that \( \sigma(i) \) is the \((\ell_i(\sigma) + 1)\)-th smallest element of the set \([n] \setminus \{\sigma(1), \sigma(2), \ldots, \sigma(i-1)\}\).

(b) For any \( \sigma \in S_n \), prove that \((\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) \in G\).

(c) Prove that the map

\[
S_n \to G, \\
\sigma \mapsto (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))
\]

is bijective.

(d) Show that \( \ell(\sigma) = \ell_1(\sigma) + \ell_2(\sigma) + \cdots + \ell_n(\sigma) \) for each \( \sigma \in S_n \).

(e) Show that

\[
\sum_{\sigma \in S_n} x^{\ell(\sigma)} = (1 + x) \left( 1 + x + x^2 \right) \cdots \left( 1 + x + x^2 + \cdots + x^{n-1} \right)
\]

(an equality between polynomials in \( x \)). (If \( n \leq 1 \), then the right hand side of this equality is an empty product, and thus equals 1.)

Note that the \( n \)-tuple \((\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))\) is known as the Lehmer code of the permutation \( \sigma \).

0.5. Permutations as composed transpositions

Recall a basic notation regarding permutations, which we shall now extend:

Definition 0.3. Let \( n \in \mathbb{N} \). Let \( i \) and \( j \) be two distinct elements of \([n]\). We let \( t_{i,j} \) be the permutation in \( S_n \) which switches \( i \) with \( j \) while leaving all other elements of \([n]\) unchanged. Such a permutation is called a transposition.

Let us furthermore set \( t_{i,i} = \text{id} \) for each \( i \in [n] \). Thus, \( t_{i,j} \) is defined even when \( i \) and \( j \) are not distinct.

Exercise 6. Let \( n \in \mathbb{N} \). Let \( \sigma \in S_n \).

(a) Prove that there is a unique \( n \)-tuple \((i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n] \) such that

\[
\sigma = t_{i_1,j_1} \circ t_{i_2,j_2} \circ \cdots \circ t_{i_n,j_n}.
\]
Consider this \(n\)-tuple \((i_1, i_2, \ldots, i_n)\). Define the relation \(\sim\) and the \(\sim\)-equivalence classes \(E_1, E_2, \ldots, E_m\) as in Exercise 7 on homework set \#7 (for \(X = [n]\)). (Thus, \(m\) is the number of cycles in the cycle decomposition of \(\sigma\).) Prove that \(m\) is the number of all \(k \in [n]\) satisfying \(i_k = k\).

### 0.6. Another partition identity

Recall the following:

**Definition 0.4.** Let \(n \in \mathbb{Z}\). A partition of \(n\) means a finite list \((i_1, i_2, \ldots, i_k)\) of positive integers satisfying

\[
i_1 \geq i_2 \geq \cdots \geq i_k \quad \text{and} \quad i_1 + i_2 + \cdots + i_k = n.
\]

**Exercise 7.** Let \(n \in \mathbb{N}\) and \(p \in \mathbb{N}\). Let \(a\) be the number of all partitions \((i_1, i_2, \ldots, i_k)\) of \(n\) satisfying \(k \geq p\) and \(i_1 = i_2 = \cdots = i_p\). Let \(b\) be the number of all partitions \((i_1, i_2, \ldots, i_k)\) of \(n\) such that all of \(i_1, i_2, \ldots, i_k\) are \(\geq p\). Prove that \(a = b\).

**Example 0.5.** Let \(n = 9\) and \(p = 3\). Then, the partitions counted by \(a\) in Exercise 7 are

\[(3, 3, 3),\quad (2, 2, 2, 2, 1),\quad (2, 2, 2, 1, 1, 1),\quad (1, 1, 1, 1, 1, 1, 1, 1, 1).\]

Meanwhile, the partitions counted by \(b\) in Exercise 7 are

\[(9),\quad (6, 3),\quad (5, 4),\quad (3, 3, 3).\]

Thus, \(a = 4\) and \(b = 4\) in this case.

Further reading on partitions includes:


The Wikipedia articles on partitions, the pentagonal number theorem and Ramanujan’s congruences are also useful. That said, none of these is necessary for the above exercise.