0.1. One last binomial sum

Exercise 1. Let \( n \in \mathbb{N} \). Prove that
\[
\sum_{k=0}^{n} \binom{-2}{k} = (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor.
\]

0.2. The Cartesian product of two permutations

We have defined the sign of a permutation of \([n]\) for an \( n \in \mathbb{N} \). But we can, more generally, define the sign of a permutation of any finite set. This would be difficult to define directly; instead, we define it by reducing it to a permutation of \([n]\) as follows:

Definition 0.1. Let \( X \) be a finite set. We want to define the sign of any permutation of \( X \).

Fix a bijection \( \phi : X \to [n] \) for some \( n \in \mathbb{N} \). (Such a bijection always exists.)

For every permutation \( \sigma \) of \( X \), set
\[
(-1)^\sigma = (-1)^{\phi \circ \sigma \circ \phi^{-1}}.
\]

Here, the right hand side is well-defined because \( \phi \circ \sigma \circ \phi^{-1} \) is a permutation of \([n]\).

It is not hard to check (see [Grinbe16, Exercise 4.12 (a)]) that \((-1)^\sigma\) depends only on the permutation \( \sigma \) of \( X \), but not on the bijection \( \phi \). (In other words, for a given \( \sigma \), any two different choices of \( \phi \) will lead to the same \((-1)^\sigma\).

This allows us to define the sign of the permutation \( \sigma \) to be \((-1)^\sigma\) for any choice of bijection \( \phi : X \to [n] \). We denote this sign simply by \((-1)^\sigma\). (When \( X = [n] \), then this sign is clearly the same as the sign \((-1)^\sigma\) we defined before, because we can pick the bijection \( \phi = \text{id} \).

(In contrast, we could not define the length \( \ell(\sigma) \) of a permutation \( \sigma \) of \( X \), because different bijections \( \phi \) can lead to different values of \( \ell(\phi \circ \sigma \circ \phi^{-1}) \).)

The sign of a permutation \( \sigma \) of a finite set \( X \) has the following properties (see [Grinbe16, Exercise 4.12]):

- The permutation \( \text{id} : X \to X \) satisfies \((-1)^\text{id} = 1 \).
- We have \((-1)^{\sigma \circ \tau} = (-1)^\sigma \cdot (-1)^\tau\) for any two permutations \( \sigma \) and \( \tau \) of \( X \).
Exercise 2. Let $U$ and $V$ be two finite sets. Let $\sigma$ be a permutation of $U$. Let $\tau$ be a permutation of $V$. We define a permutation $\sigma \times \tau$ of the set $U \times V$ by setting

$$(\sigma \times \tau) (a, b) = (\sigma (a), \tau (b))$$

for every $(a, b) \in U \times V$.

(a) Prove that $\sigma \times \tau$ is a well-defined permutation.
(b) Prove that $\sigma \times \tau = (\sigma \times \text{id}) \circ (\text{id} \times \tau)$.
(c) Prove that $(-1)^{\sigma \times \tau} = ((-1)^\sigma)^{|V|} ((-1)^\tau)^{|U|}$. (All the signs here are well-defined due to Definition 0.1.)

(Can you find a slick proof for part (c) that involves no endless stream of trivial lemmas?)

0.3. Non-cut vertices I

See solutions to Spring 2017 Math 5707 homework set #2 (specifically, Section 0.1) for definitions of simple graphs, multigraphs, digraphs and multidigraphs. Note, in particular, that all of these are assumed to be finite (i.e., they have finitely many vertices and finitely many edges).

Recall that a multigraph is defined to be a triple $(V, E, \varphi)$, where $V$ and $E$ are two finite sets and $\varphi$ is a map $E \to \mathcal{P}_2 (V)$ (sending each “edge” $e \in E$ to an unordered pair of two distinct “vertices”). The elements of $V$ are called the vertices of the multigraph; the elements of $E$ are called its edges.

Definition 0.2. Let $G = (V, E, \varphi)$ and $G' = (V', E', \varphi')$ be two multigraphs. We say that $G'$ is a subgraph of $G$ if and only if $V' \subseteq V$, $E' \subseteq E$ and $(\varphi' (e) = \varphi (e)$ for each $e \in E'$).

Thus, a subgraph of a multigraph $G$ is simply a multigraph obtained from $G$ by removing some vertices and some edges provided that for each vertex we remove, all edges containing that vertex are also removed. For example, the 2-vertex graph 1 — 2 has 5 subgraphs: itself; the subgraph obtained by removing the edge (but leaving both vertices intact); the two subgraphs obtained by removing one vertex (along with the edge); and finally the subgraph obtained by removing everything.

Definition 0.3. Let $G = (V, E, \varphi)$ be a multigraph. Let $v \in V$ be a vertex. Then, $G \setminus v$ shall denote the subgraph $(V \setminus \{v\}, E', \varphi |_{E'})$ of $G$, where $E'$ is the set of all edges $e \in E$ that don’t contain the vertex $v$. In other words, $G \setminus v$ is the subgraph of $G$ obtained by removing the vertex $v$ and all edges containing $v$.

For example, if $G$ is the 3-vertex graph 1 — 2 — 3, then $G \setminus 1$ is the 2-vertex graph 2 — 3, whereas $G \setminus 2$ is the 2-vertex graph 1 — 3.

\^{1}Some\” may mean \“none\”, and may also mean \“all\” (as well as anything inbetween).
Definition 0.4. Let $G = (V, E, \varphi)$ be a multigraph. A vertex $v \in V$ is said to be non-cut if the multigraph $G \setminus v$ is connected or has no vertices.

For example, if $G$ is the 3-vertex graph $\{1, 2, 3\}$, then the non-cut vertices of $G$ are 1 and 3.

Definition 0.5. Let $G$ be a multigraph. Let $(v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k)$ be a walk in $G$. Then, the length of this walk is defined to be $k$ (that is, the number of edges).

Definition 0.6. Let $G = (V, E, \varphi)$ be a connected multigraph. Let $v \in V$ and $w \in V$ be two vertices. Then, $d(v, w)$ (the distance between $v$ and $w$) is defined as the smallest length of a path from $v$ to $w$. (This is also the smallest length of a walk from $v$ to $w$, because every walk from $v$ to $w$ can be trimmed down to a path of the same or smaller length.)

Exercise 3. Let $G = (V, E, \varphi)$ be a connected multigraph. Let $v \in V$ be any vertex.

(a) Pick any $w \in V$ such that $d(v, w)$ is maximum (among all $w \in V$). Prove that $w$ is a non-cut vertex of $G$.

(b) Let $n = |V|$. Prove that $\sum_{u \in V} d(v, u) \leq \binom{n}{2}$.

Exercise 3(a) is particularly important, as it guarantees that any connected multigraph with at least one vertex has a non-cut vertex. This allows proving properties of connected multigraphs by induction on the number of vertices.

Note that the inequality in Exercise 3(b) is sharp (i.e., equality can hold): If $V$ is the $n$-vertex graph $\{1, 2, 3, \ldots, n\}$ and if $v = 1$, then $\sum_{u \in V} d(v, u) = 0 + 1 + \cdots + (n - 1) = \binom{n}{2}$.

0.4. Non-cut vertices II: subgraphs

Exercise 4. Let $G$ be a connected multigraph. Let $H$ be a connected subgraph of $G$. Prove that the number of non-cut vertices of $H$ is $\leq$ to the number of non-cut vertices of $G$.

0.5. When do transpositions generate all permutations?

Exercise 5. Let $G = (V, E, \varphi)$ be a connected multigraph.

For each $e = \{u, v\} \in \mathcal{P}_2(V)$, we let $t_e$ be the permutation of $V$ that switches $u$ with $v$ while leaving all other elements of $V$ unchanged.

An $E$-transposition shall mean a permutation of the form $t_e$ for some $e \in \varphi(E)$.

Prove that every permutation of $V$ can be written as a composition of some $E$-transpositions.
[You are allowed to use the result of Exercise 3(a) here even if you have not solved that exercise.]

Note that Exercise 5 generalizes Exercise 3 on Math 4990 homework set #7, because the simple graph $([n], \{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\})$ (for $n > 0$) is connected.

Exercise 5 also generalizes the fact that every permutation of $[n]$ can be written as a composition of simple transpositions $s_i = t_{i, i+1}$, because the simple graph $([n], \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}\})$ (for $n > 0$) is connected.

Exercise 5 also has a converse: If $G = (V, E, \phi)$ is a multigraph such that every permutation of $V$ can be written as a composition of some $E$-transpositions, then $G$ is connected or $V$ is empty. This is not hard to check.

### 0.6. Watersheds in digraphs
Example 0.7. Consider the following digraph:

Imagine a game chip placed initially at the vertex 1. The chip is allowed to move along the arcs of the digraph (from source to target). For example, the chip can first move along the arc \((1, 2)\) to 2, then along the arc \((2, 3)\) to 3, then along the arc \((3, 5)\) to 5. Once it arrives at 5, it can no longer move, because there are no arcs with source 5. We say that 5 is a sink for this reason (see Exercise 6 below for the precise definition).

Alternatively, the sink could have moved along the arc \((1, 2)\) to 2, then along the arc \((2, 6)\) to 6, then along the arc \((6, 7)\) to 7. At this point it would again be stuck, since 7 is a sink.

Thus, the chip can get stuck in two different sinks, depending on the path it takes. (It will always get stuck in some sink, because our digraph has no cycles.)

Now, consider the following digraph:

This time, any chip starting at any given vertex will necessarily get stuck at the same sink no matter what path it takes (either the sink 1, if it started at one of the vertices 1, 2, 3, 4, 5, 6, 7; or the sink 9, if it started at one of the vertices 8, 9, 10). How can we show this without checking all possible paths?

One criterion, which is clearly necessary, is that there are no “watershed vertices”: i.e., there is no vertex \(u\) from which the chip can take two different arcs
Exercise 6. Let $D$ be a multidigraph having no cycles. A vertex $v$ of $D$ is said to be a sink if there is no arc of $D$ with source $v$.

If $u$ and $v$ are any two vertices of $D$, then:

- we write $u \rightarrow v$ if and only if $D$ has an arc with source $u$ and target $v$;
- we write $u \ast \rightarrow v$ if and only if $D$ has a path from $u$ to $v$.

The so-called no-watershed condition says that for any three vertices $u$, $v$ and $w$ of $D$ satisfying $u \rightarrow v$ and $u \rightarrow w$, there exists a vertex $t$ of $D$ such that $v \ast \rightarrow t$ and $w \ast \rightarrow t$.

Assume that the no-watershed condition holds. Prove that for each vertex $p$ of $D$, there exists a unique sink $q$ of $D$ such that $p \ast \rightarrow q$.

[Hint: Induction on the “height” of $p$ (that is, the length of a longest path starting at $p$).]

0.7. Acyclic orientations and source pushing

Roughly speaking, an orientation of a multigraph $G$ is a way to assign to each edge of $G$ a direction (thus making it an arc). If the resulting multidigraph has no cycles, then this orientation will be called acyclic. A rigorous way to state this definition is the following:

Definition 0.8. Let $G = (V, E, \psi)$ be a multigraph.

(a) An orientation of $G$ is a map $\phi : E \rightarrow V \times V$ such that each $e \in E$ has the following property: If we write $\phi(e)$ in the form $\phi(e) = (u, v)$, then $\psi(e) = \{u, v\}$.

(b) An orientation $\phi$ of $G$ is said to be acyclic if and only if the multidigraph $(V, E, \phi)$ has no cycles.

Example 0.9. Let $G = (V, E, \psi)$ be the following multigraph:

\[
\begin{array}{c}
\begin{array}{c}
1 \\
\uparrow \quad \downarrow \quad \uparrow \quad \downarrow \\
\quad a \\
\quad \downarrow \\
3
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
2 \\
\uparrow \quad \downarrow \quad \uparrow \quad \downarrow \\
\quad b \\
\quad \downarrow \\
3
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
3 \\
\uparrow \quad \downarrow \quad \uparrow \quad \downarrow \\
\quad c
\end{array}
\end{array}
\]
Then, the following four maps $\phi$ are orientations of $G$:

- the map sending $a$ to $(1,2)$, sending $b$ to $(1,2)$, sending $c$ to $(3,2)$, and sending $d$ to $(1,3)$;
- the map sending $a$ to $(2,1)$, sending $b$ to $(1,2)$, sending $c$ to $(3,2)$, and sending $d$ to $(3,1)$;
- the map sending $a$ to $(1,2)$, sending $b$ to $(1,2)$, sending $c$ to $(2,3)$, and sending $d$ to $(1,3)$;
- the map sending $a$ to $(1,2)$, sending $b$ to $(1,2)$, sending $c$ to $(2,3)$, and sending $d$ to $(3,1)$.

Here are the multidigraphs $(V,E,\phi)$ corresponding to these four maps (in the order mentioned):

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Only the first and the third of these orientations $\phi$ are acyclic (since only the first and the third of these multidigraphs have no cycles).

**Definition 0.10.** Let $G = (V,E,\psi)$ be a multigraph. Let $\phi$ be an orientation of $G$.

A vertex $v \in V$ is said to be a source of $\phi$ if no arc of the multidigraph $(V,E,\phi)$ has target $v$. Exercise 6 (a) on Math 5707 (Spring 2017) homework set #5 shows that if $\phi$ is acyclic and if $V \neq \emptyset$, then there exists a source of $\phi$.

If $v$ is a source of $\phi$, then we can define a new orientation $\phi'$ of $G$ as follows:

- For each $e \in E$ satisfying $v \in \psi(e)$, we set $\phi'(e) = (u,v)$, where $u$ is chosen such that $\phi(e) = (v,u)$.
- For all other $e \in E$, we set $\phi'(e) = \phi(e)$.

(Roughly speaking, this simply means that $\phi'$ is obtained by $\phi$ by reversing the directions of all edges that contain $v$.) We say that this new orientation $\phi'$ is obtained from $\phi$ by pushing the source $v$. 
Example 0.11. Let $G = (V, E, \psi)$ be the following multigraph:

Consider the orientation $\phi$ of $G$ for which the multidigraph $(V, E, \phi)$ looks as follows:

(Formally speaking, this is the orientation $\phi$ that sends the edges $a, b, c, d, e, f$ to the pairs $(1, 2), (3, 2), (1, 4), (3, 5), (4, 5), (5, 4)$, respectively.)

This orientation $\phi$ has two sources 1 and 3. We can transform this orientation by pushing the source 1; this results in the following orientation $\phi'$ (shown here by drawing the multidigraph $(V, E, \phi')$):

This new orientation $\phi'$ has a single source, 3. If we push this source, we obtain a new orientation $\phi''$, which looks as follows (again, represented by the multidigraph $(V, E, \phi'')$):

This orientation $\phi''$, in turn, has a single source, 2. If we push this source, we
obtain a new orientation $\phi'''$, which looks as follows (again, represented by the multidigraph $(V, E, \phi''')$):

![Diagram](image)

This orientation $\phi'''$ has no sources, and thus cannot be transformed any further by pushing sources.

The preceding example suggests some questions: For example, given an orientation of a multigraph, can we keep pushing sources indefinitely, or will we eventually end up at an orientation that has no more sources? The following is easy to see:

**Proposition 0.12.** Let $\phi$ be an acyclic orientation of a multigraph $G = (V, E, \psi)$. Let $v$ be a source of $\phi$. Then, the orientation obtained from $\phi$ by pushing the source $v$ is again acyclic.

This proposition shows that if we start with an acyclic orientation of a multigraph (with at least one vertex), then we can keep pushing sources indefinitely (since the orientation always remains acyclic, and thus there always will be sources to push). The next exercise (specifically, Exercise 7(c)) yields a converse (for connected multigraphs): If we can keep pushing sources indefinitely (or, even, if we can keep pushing sources for more than $\left\lceil \frac{n}{2} \right\rceil$ times in a row), then our orientation must have been acyclic.

**Exercise 7.** Let $G = (V, E, \psi)$ be a connected multigraph. Set $n = |V|$.

Let $(\phi_0, \phi_1, \ldots, \phi_k)$ be a sequence of orientations of $G$, and let $(v_1, v_2, \ldots, v_k)$ be a sequence of vertices of $G$ such that for each $i \in \{1, 2, \ldots, k\}$, the orientation $\phi_i$ is obtained from $\phi_{i-1}$ by pushing the source $v_i$ (in particular, this is saying that $v_i$ is a source of $\phi_{i-1}$).

(a) Prove that if $u$ and $w$ are two mutually adjacent vertices of $G$, then between any two consecutive appearances of $u$ in the sequence $(v_1, v_2, \ldots, v_k)$, the vertex $w$ must appear at least once.

Now, assume that $k > \left( \frac{n}{2} \right)$.

(b) Prove that each vertex of $G$ appears at least once in the sequence $(v_1, v_2, \ldots, v_k)$.

(c) Prove that the orientations $\phi_0, \phi_1, \ldots, \phi_k$ are acyclic.
[Hint: For part (b), assume that some vertex \( v \) does not appear in the sequence \((v_1, v_2, \ldots, v_k)\). Then, argue that any vertex \( u \in V \) appears at most \( d(v, u) \) times in this sequence, using part (a). Then apply Exercise 3(b). For part (c), first argue that any cycle existing in one of the orientations \( \phi_0, \phi_1, \ldots, \phi_k \) would automatically exist in all of these orientations.]

[You may use Exercise 3(b) even if you have not solved this exercise.]

Exercise 8. Let \( G = (V, E, \psi) \) be a connected multigraph.

Fix a vertex \( v \in V \).

If \( \phi \) and \( \phi' \) are two orientations of \( G \), then we write \( \phi \xrightarrow{v} \phi' \) if and only if \( \phi' \) is obtained from \( \phi \) by repeatedly pushing sources without ever pushing the source \( v \). (More rigorously: We write \( \phi \xrightarrow{v} \phi' \) if and only if there exist a sequence \((\phi_0, \phi_1, \ldots, \phi_k)\) of orientations of \( G \) and a sequence \((v_1, v_2, \ldots, v_k)\) of vertices of \( G \) distinct from \( v \) such that for each \( i \in \{1, 2, \ldots, k\} \), the orientation \( \phi_i \) is obtained from \( \phi_{i-1} \) by pushing the source \( v_i \) (in particular, this is saying that \( v_i \) is a source of \( \phi_{i-1} \)), and such that \( \phi_0 = \phi \) and \( \phi_k = \phi' \).)

If \( \phi \) is an orientation of \( G \), then we say that \( \phi \) is \( v \)-fleeing if \( \phi \) has no source other than \( v \). (Note that \( \phi \) may or may not have \( v \) as a source.)

For any orientation \( \phi \) of \( G \), prove that there is a unique \( v \)-fleeing orientation \( \phi' \) such that \( \phi \xrightarrow{v} \phi' \).

[Hint: Consider a new multidigraph \( O_v \) whose vertices are the orientations of \( G \), and which has an arc from an orientation \( \phi \) to an orientation \( \phi' \) if and only if \( \phi' \) can be obtained from \( \phi \) by pushing a source different from \( v \). Use Exercise 7(b) to argue that this multidigraph \( O_v \) has no cycles, and then use Exercise 6.]

[You may use both exercises mentioned in the hint without solving them.]

References
