Exercise 1. Let $n \in \mathbb{N}$.
(a) Find the number of all triples $(A, B, C)$ of subsets of $[n]$ satisfying $A \cup B \cup C = [n]$ and $A \cap B \cap C = \emptyset$.
(b) Find the number of all triples $(A, B, C)$ of subsets of $[n]$ satisfying $B \cap C = C \cap A = A \cap B$.

Recall that if $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $\text{sur}(n,k)$ denotes the number of surjections $[n] \to [k]$, and $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ denotes the Stirling number of the 2nd kind (defined as $\text{sur}(n,k) / k!$).

Exercise 2. Let $n$ be a positive integer. Let $k \in \mathbb{N}$.
(a) Prove that $\text{sur}(n,k) = k \sum_{i=0}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^{n-1}$.
(b) Prove that $\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \sum_{i=0}^{k} (-1)^{k-i} \frac{i^n}{i! (k-i)!}$.

Exercise 3. A set $S$ of integers is said to be 2-lacunar if every $i \in S$ satisfies $i + 1 \notin S$ and $i + 2 \notin S$. (That is, any two distinct elements of $S$ are at least a distance of 3 apart on the real axis.) For example, $\{1, 5, 8\}$ is 2-lacunar, but $\{1, 5, 7\}$ is not.

For any $n \in \mathbb{N}$, we let $h(n)$ denote the number of all 2-lacunar subsets of $[n]$.
(a) Prove that $h(n) = h(n-1) + h(n-3)$ for each $n \geq 3$.
(b) Prove that $h(n) = \sum_{k \in \mathbb{N}; \ 2k \leq n+2} \binom{n+2-2k}{k}$ for each $n \in \mathbb{N}$.

Exercise 4. A set $S$ of integers is said to be shadowed if it has the following property: Whenever an odd integer $i$ belongs to $S$, the next integer $i+1$ must also belong to $S$. (For example, $\emptyset$, $\{2, 4\}$ and $\{1, 2, 5, 6, 8\}$ are shadowed, but $\{1, 5, 7\}$ is not, since 1 belongs to $\{1, 5, 6\}$ but 2 does not.)

Let $n \in \mathbb{N}$ be even. How many shadowed subsets of $[n]$ exist?

Exercise 5. Let $n$ and $k$ be positive integers. A k-smord will mean a $k$-tuple $(a_1, a_2, \ldots, a_k) \in [n]^k$ such that no two consecutive entries of the $k$-tuple are equal (i.e., we have $a_i \neq a_{i+1}$ for all $i \in [k-1]$). For example, $(3, 1, 3, 2)$ is a 4-smord (when $n \geq 3$), but $(1, 3, 3, 2)$ is not.

Compute the number of all $k$-smords.
Exercise 6. This continues Exercise 7 from homework set 2.

Let $n$ be a positive integer. Let $X$ be a set.

We define a map $c : X^n \to X^n$ by

$$c(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1)$$

for all $(x_1, x_2, \ldots, x_n) \in X^n$.

(In other words, the map $c$ transforms any $n$-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ by “rotating” it one step to the left, or, equivalently, moving its first entry to the last position.)

For two $n$-tuples $x$ and $y$, we say that $x \sim y$ if there exists some $k \in \mathbb{N}$ such that $y = c^k(x)$. (For example, $(1, 5, 2, 4) \sim (2, 4, 1, 5)$, because $(2, 4, 1, 5) = c^2(1, 5, 2, 4)$.)

(a) Prove that $\sim$ is an equivalence relation, i.e., is reflexive, transitive and symmetric. (For example, symmetry boils down to showing that if there exists some $k \in \mathbb{N}$ satisfying $y = c^k(x)$, then there exists some $\ell \in \mathbb{N}$ satisfying $x = c^\ell(y)$.)

(b) An $n$-necklace (over $X$) shall mean a $\sim$-equivalence class. We denote the $\sim$-equivalence class of a tuple $x \in X^n$ by $[x]_\sim$.

Let $x \in X^n$ be an $n$-tuple. Let $m$ be the smallest nonzero period of the $n$-tuple $x \in X^n$.

Prove that $[x]_\sim = \{c^0(x), c^1(x), \ldots, c^{m-1}(x)\}$.

(c) Show that the $m$ tuples $c^0(x), c^1(x), \ldots, c^{m-1}(x)$ are distinct. Conclude that $|[x]_\sim| = m$. 