0.1. Idempotent maps

If $S$ is a set, then a map $f : S \rightarrow S$ is said to be idempotent if and only if $f \circ f = f$. For instance, the map $[3] \rightarrow [3]$ sending 1, 2, 3 to 1, 3, 3 (respectively) is idempotent.

**Exercise 1.** Let $n \in \mathbb{N}$.

(a) Prove that a map $f : [n] \rightarrow [n]$ is idempotent if and only if $f(y) = y$ for every $y$ in the image of $f$.

(b) Prove that the number of idempotent maps $[n] \rightarrow [n]$ is $\sum_{k=0}^{n} \binom{n}{k} k^{n-k}$.

(c) Prove that the number of idempotent maps $[n] \rightarrow [n]$ has the form $an + 1$ for some $a \in \mathbb{N}$. (Of course, $a$ will depend on $n$.)

[Hint: When is $\binom{n}{k} k^{n-k}$ divisible by $n$?]

0.2. Fixed points

**Exercise 2.** Let $S$ be a finite set. For any map $f : S \rightarrow S$, we let $\text{Fix } f$ denote the set of all fixed points of $f$. (That is, $\text{Fix } f = \{ s \in S \mid f(s) = s \}$.)

(a) Prove that $|\text{Fix } (f \circ g)| = |\text{Fix } (g \circ f)|$ for any two maps $f : S \rightarrow S$ and $g : S \rightarrow S$.

(b) Is it true that every three maps $f, g, h$ from $S$ to $S$ satisfy $|\text{Fix } (f \circ g \circ h)| = |\text{Fix } (g \circ f \circ h)|$?

[Hint: For (a), find a bijection.]

0.3. A binomial coefficient in a denominator

**Exercise 3.** Let $n$ and $a$ be two integers with $n \geq a \geq 1$. Prove that

$$\sum_{k=a}^{n} \frac{(-1)^k}{k} \binom{n}{k-a} = \frac{(-1)^a}{a} \binom{n}{a}.$$ 

0.4. Derangements with at most 1 descent

**Exercise 4.** Let $n \in \mathbb{N}$. How many derangements $\sigma$ of $[n]$ have at most 1 descent?

(See homework set #5 for the definitions of descents and of derangements.)
0.5. Connected permutations

**Definition 0.1.** Let \( n \) be a positive integer. A permutation \( \sigma \) of \([n]\) is said to be **connected** if and only if there exists no \( k \in [n - 1] \) such that \( \sigma([k]) = [k] \).

For example, the permutation \( \sigma \) of \([5]\) sending 1, 2, 3, 4, 5 to 2, 4, 1, 5, 3 is connected, since it satisfies
\[
\sigma([1]) = \{2\} \neq [1], \quad \sigma([2]) = \{2, 4\} \neq [2],
\]
\[
\sigma([3]) = \{2, 4, 1\} \neq [3], \quad \sigma([4]) = \{2, 4, 1, 5\} \neq [4].
\]

But the permutation \( \sigma \) of \([4]\) sending 1, 2, 3, 4 to 2, 1, 4, 3 is not connected, because it satisfies \( \sigma([2]) = [2] \).

Likewise, a permutation \( \sigma \) of \([n]\) (for \( n > 1 \)) satisfying \( \sigma(1) = 1 \) is never connected (since \( \sigma([1]) = [1] \)); the same holds for a permutation \( \sigma \) satisfying \( \sigma(n) = n \) (since \( \sigma([n - 1]) = [n - 1] \)).

**Exercise 5.** For each positive integer \( n \), let \( c_n \) denote the number of all connected permutations of \([n]\). (Thus, \( c_1 = 1, c_2 = 1 \) and \( c_3 = 3 \).)

Prove that
\[
n! = \sum_{k=1}^{n} c_k (n - k)! \quad \text{for each positive integer } n.
\]

0.6. Permutations and intervals

An **integer interval** means a set of the form \( \{a, a+1, \ldots, b\} \) for some integers \( a \) and \( b \). (If \( a > b \), then this set is understood to be empty.)

**Exercise 6.** Let \( n \in \mathbb{N} \) and \( r \in [n] \). A permutation \( \sigma \) of \([n]\) is said to be **\( r \)-friendly** if for each \( k \in \{r, r+1, \ldots, n\} \), the set \( \sigma([k]) \) is an integer interval.

(For example, the permutation \( \sigma \) of \([6]\) sending 1, 2, 3, 4, 5, 6 to 2, 4, 3, 5, 1, 6 is 3-friendly (since \( \sigma([3]) = \{2, 3, 4\} \), \( \sigma([4]) = \{2, 3, 4, 5\} \), \( \sigma([5]) = \{1, 2, 3, 4, 5\} \) and \( \sigma([6]) = \{1, 2, 3, 4, 5, 6\} \) are integer intervals), and thus also \( r \)-friendly for each \( r \geq 3 \), but not 2-friendly (since \( \sigma([2]) = \{2, 4\} \) is not an integer interval).)

Prove that the number of \( r \)-friendly permutations of \([n]\) is \( 2^{n-r}r! \).

0.7. Inverting a power series

**Exercise 7.** Find and prove an explicit formula for the coefficient of \( x^n \) in the formal power series \( \frac{1}{1 - x - x^2 + x^3} \).

**[Hint:** The standard strategy is to factor \( 1 - x - x^2 + x^3 \), then do partial fraction decomposition. But it is perfectly legitimate to guess the formula based on...\]
solving

\[
\left(1 - x - x^2 + x^3\right) \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots\right) = 1
\]

for the first few of the unknown coefficients \(b_0, b_1, b_2, \ldots\), and then prove it by multiplying out. Either option works.]