Def. A permutation $\pi$ of $\{1, 2, \ldots, n\}$ is short-legged if $\forall i \in [n]$ we have $|\pi(i) - i| \leq 1$.

Q: How many short-legged permutations are there?

Ex: For $n=3$, these are

$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$

Ex: For $n=4$, these are

$(1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 2, 4), (2, 1, 3, 4), (2, 1, 4, 3)$

Prop. 1.21. Let $n \in \mathbb{N}$. The number of short-legged permutations of $[n]$ is $f_{n+1}$.

Proof idea. Short-legged permutation of $[8]$:

$(1, \textbf{3}, 2, 4, 5, \textbf{7}, 6, \textbf{8})$

Proof idea. Short-legged permutation of $[8]$:

$(1, 2, 3, 4, 5, 6, 7, 8)$
This gives a domino tiling of a 2×8-rectangle.

Conversely:

Thus, we have a bijection

\[ \{\text{short-legged permutations of } [n]^3\} \]

\[ \rightarrow \{\text{domino tilings of } 2 \times n^3\text{-rectangle}\} \]

But Prop. 1.4 shows that the # of the latter is \( h_{n+1} \).

(Yes, this is informal.)
Counting subsets...

- An $n$-element set $S$ has $2^n$ subsets.
- An $n$-element set has $\binom{n}{k}$ $k$-elt subsets.

**Def.** A set $S$ of integers is **lacunar** if it contains no two consecutive integers (i.e., no $i \in \mathbb{Z}$ such that $i \in S$ and $i+1 \in S$).

**Q:** How many lacunar subsets does $\{n\}$ have?

**Prop. 1.22.** Let $n \in \mathbb{N}$. The number of lacunar subsets of $\{n\}$ is $f_{n+2}$.

1st Proof idea: Induction.

2nd proof idea:  
- A lacunar subset $\{2 < 4 < 7 < 9 < 13\}$ of $\{13\}$.

```
1 2 3 4 5 6 7 8 9 10 11 12 13
```

domino tiling of $2 \times 14$ rectangle
Again, a bijection.

Prop. 1.23. Let \( n \in \mathbb{N} \). The largest size of a lacunar subset of \([n]\) is \( \lceil \frac{n}{27} \rceil \), where \( \lceil x \rceil \) denotes the smallest integer \( \geq x \) ("ceiling" of \( x \), aka rounding up \( x \)).

Proof. The lacunar subset \( \{1 < 3 < 5 < \ldots < (n \text{ or } n-1)\} \) has size \( \lceil \frac{n}{27} \rceil \). So we only need to show that no higher size is possible.

Assume the contrary. So \( \exists \) lacunar subset \( S \) of \([n]\) with \( |S| > \lceil \frac{n}{27} \rceil \). Hence \( |S| \geq \lceil \frac{n}{27} \rceil + 1 > \frac{n+1}{2} \).

Thus, \( 2|S| > n+1 \).

Let \( S^+ \) be the set \( \{s+1 \mid s \in S\} \).

(If \( S = \{2, 4, 7, 10\} \), then \( S^+ = \{3, 5, 8, 11\} \).)

Then, \( S \) and \( S^+ \) are two subsets of \([n+1]\). Thus,

\[ |S \cup S^+| \leq n+1. \]

But \( S \) and \( S^+ \) are disjoint (since \( S \) is lacunar), so \( |S \cup S^+| = |S| + |S^+| = |S| + |S| \) (since \( |S^+| = |S| \)).
\[ = 2|S| > n+1. \]

The previous 2 inequalities contradict each other. \( \square \)

2. Induction

We'll see many versions of induction. We start with the simplest one:

2.1. Standard induction \((0, n \mapsto n+1)\).

Induction principle 2.1. For each \( n \in \mathbb{N} \), let \( \#(n) \) be a statement.

Assume that

- \( \#(0) \) is true,
- for each \( n \in \mathbb{N} \): if \( \#(n) \) is true, then \( \#(n+1) \) is true.

Then, \( \#(n) \) is true \( \forall n \in \mathbb{N} \).

Example 2.2 (Towers of Hanoi): Given a wooden board, 3 pegs and \( n \) disks with little huts, so they can fit on the pegs.
The disks have different sizes, originally all disks are wrapped around peg 1, with the largest disk at the very bottom, the next-largest 2 step further up, ... In a single step, you can take the topmost disk from a peg & move it to a different peg,
provided that it does not get placed on a smaller disk.

The goal is to move all disks to peg 3.

We call this "solving n-disk ToH".

Q: Can we always solve n-disk ToH?

In how many steps?

ex: n = 3:

Thus, the 3-disk ToH can be solved in 7 steps.
n = 2:
\[
\begin{align*}
\frac{1}{2} & \Rightarrow \frac{1}{1} \Rightarrow \frac{1}{2} \\
\frac{1}{2} & \Rightarrow \frac{1}{1} \Rightarrow \frac{1}{2}
\end{align*}
\]

The 2-disk ToH needs 3 steps.
The 1-disk ________ 1 step.
The 0-disk ________ 0 steps.

Prop. 2.3. Let n \in \mathbb{N},

We can solve the n-disk ToH in \(2^n - 1\) steps

and cannot solve it in fewer steps.

Proof. Induction on n; i.e. use principle 2.1.

Let \(\mathcal{A}(n)\) = "We can solve the n-disk ToH in \(2^n - 1\) steps

and cannot solve it in fewer steps".

Then, \(\mathcal{A}(0)\) is true.

Now, fix \(n \in \mathbb{N}\). We want to prove that if \(\mathcal{A}(n)\) is true,

then \(\mathcal{A}(n+1)\) is true.
Assume $A(n)$ is true.

We must prove that $A(n+1)$ is true, i.e., that the $(n+1)$-disk ToH can be solved in $2^{n+1} - 1$ steps but not in fewer.

To see that it can be solved in $2^{n+1} - 1$ steps, proceed as follows:

• First, move the disks 1, 2, ..., $n$ to peg 2.
This can be done in $2^n - 1$ steps (as it is just an $n$-disk ToH, so $A(n)$ tells us it can be solved in $2^n - 1$ steps).

• Then, move disk $n+1$ to peg 3 (in 1 step).

• Then, move the disks 1, 2, ..., $n$ to peg 3.
This can be done in $2^n - 1$ steps (as it is again just an $n$-disk ToH).

Together, these are $(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1$ steps.
Why not in fewer?
Assume that you have some sequence $S$ of steps solving the $(n+1)$-disk ToH.

Somewhere in $S$, the disk $n+1$ must get moved.

Let $P$ be the first time it moves, and $Q$ the last time.

At step $P$, it moves from peg 1 to peg $k \in \{2, 3\}$.

At that time, and peg $k$ is empty after the move.

Thus, the remaining peg must contain the disks $1, 2, \ldots, n$.

Hence, before step $P$, an $n$-disk ToH has been solved.

Thus, by $f(n)$, at least $2^n - 1$ steps were made.

A similar argument shows that at step $Q$, the disks $1, 2, \ldots, n$ all are on the same peg $\neq 3$.

Thus, after step $Q$, we still need to solve an $n$-disk ToH.

This again, takes $\geq 2^n - 1$ steps.
Thus, we have made at least
\[
(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1
\]
steps up to \( p \) \( \frac{\text{step } p}{\text{steps after } q} \) steps in \( S \). So at least \( 2^{n+1} - 1 \) steps are needed.

Thus, \( \mathcal{A}(n+1) \) is proven.

So principle 2.1 says: \( \mathcal{A}(n) \) is true \( \forall n \in \mathbb{N} \).

\[ \square \]

Remark: Moreover,
- there is exactly one sequence of \( 2^n - 1 \) steps solving the \( n \)-disk ToH.

Remark: The shortest solution for ToH with \( >3 \) pegs is unknown.

[Lele Me, §16.4.2].
Example 2.4. For positive integers \( p \) and \( q \), a mutilated \( p \times q \)-cheesboard is a \( p \times q \)-rectangle with one square missing.

Examples:

An \( L \)-tile is a "tile" of the form

or

build of 3 \( 1 \times 1 \)-rectangles.

Prop. 2.5. [ZeLeMe, 25.1.5]. Let \( n \in \mathbb{N} \).

Any mutilated \( 2^n \times 2^n \)-cheesboard can be tiled with \( L \)-tiles.
Proof. Use induction, i.e. apply Principle 2.1.

\( \mathcal{A}(n) = \) (any mutilated \( 2^n \times 2^n \)-chessboard can be tiled with \( L \)-tiles).

Then, \( \mathcal{A}(0) \) is true, since any mutilated \( 2^0 \times 2^0 \)-chessboard is empty.

Now, fix \( n \in \mathbb{N} \), and assume \( \mathcal{A}(n) \) is true. Fix a mutilated \( 2^{n+1} \times 2^{n+1} \)-chessboard \( C \).

Split \( C \) along the vertical axes of symmetry.

Get 4 \( 2^n \times 2^n \)-chessboards \( C_{NW}, C_{NE}, C_{SW}, C_{SE} \), one of which is mutilated.
Assume WLOG that $C_{sw}$ is mutilated.
Thus, $C_{sw}$ can be tiled with L-tiles (by $\delta(n)$),

\[\begin{array}{|c|c|}
\hline
C_{NW} & C_{NE} \\
\hline
C_{SW} & C_{SE} \\
\hline
\end{array}\]

Remove three corner-squares, one for

Remove one corner-square from each of $C_{NW}, C_{NE}, C_{SE}$ in such a way that these corners form an L-tile.

Then, $C_{NW}, C_{NE}, C_{SE}$ can be tiled. Finally, $\delta(n+1)$ is true.