Proof of Thm. 1.12: We are in one of the following cases:

Case 1: \( k \in \{1, 2, 3, \ldots \} \).
Case 2: \( k = 0 \).
Case 3: \( k \in \mathbb{N} \).

Case 3 is easy: if \( k \in \mathbb{N} \), then \( k-1 \in \mathbb{N} \), so \( \binom{n-1}{k-1} = 0 \), but also \( \binom{n}{k} = 0 \) and \( \binom{n-1}{k} = 0 \), so we have to prove \( 0 = 0 + 0 \).

Case 2: Assume \( k = 0 \). Then \( \binom{n}{k} = \binom{n}{0} = \frac{(\text{empty product})}{0!} = \frac{1}{1} = 1 \), and similarly \( \binom{n-1}{k-1} = 1 \). Also \( \binom{n-1}{k-1} = \binom{n-1}{n-k} = 0 \).

So we must prove \( 1 = 0 + 1 \).

Case 1: \( k \in \{1, 2, 3, \ldots \} \). Thus,

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)(n-2) \cdots (n-k+1)}{(k-1)!} + \frac{(n-1)(n-2) \cdots (n-k)}{k!}
\]

\[
= \frac{(n-1)(n-2) \cdots (n-k+1)}{(k-1)!} \left(1 + \frac{n-k}{k} \right)
\]
\[ (\text{since } k! = k \cdot (k-1)! \text{ }) \]

\[ = \frac{(n-1)(n-2)\cdots(n-k+1)}{(k-1)!} \cdot \frac{n}{k} \]

\[ = \frac{n(n-1)\cdots(n-k+1)}{k! (n-k)!} = \binom{n}{k} \]

\text{Thm. 1.19: Let } n \in \mathbb{N}, \ k \in \mathbb{N} \text{ be such that } n \geq k. \text{ Then,}

\[ \binom{n}{k} = \frac{n!}{k! (n-k)!} \]

\text{Proof.}

\[ \binom{n}{k} \cdot (n-k)! = \frac{n(n-1)\cdots(n-k+1)}{k!} (n-k)! = \frac{n(n-1)\cdots 1}{k!} = \frac{n!}{k!} \]
Proof of Thm. 1.13:

**Case 1:** \( 0 \leq k \leq n \).

**Case 2:** \( k < 0 \).

**Case 3:** \( k > n \).

In Case 1, Thm. 1.49 yields

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

and

\[
\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{k}
\]

In Case 2: \( k < 0 \) so \( n-k > n \). Hence, Prop 1.112 yields (applied to \( n-k \) instead of \( k \))

\[
\binom{n}{n-k} = 0.
\]

But \( k < 0 \) yields \( \binom{n}{k} = 0 \). Thus, \( \binom{n}{k} = 0 = \binom{n}{n-k} \).

In Case 3: similar. □
Prop. 1.20. \( \forall k \in \mathbb{Z}, \) we have \( \binom{0}{k} = [k=0]. \)

Here, we are using the Iverson bracket notation:

**Def.** For any statement \( \phi \), we let \([\phi]\) be the **truth value** of \( \phi \), defined to be \( 1 \) if \( \phi \) is true; \( 0 \) if \( \phi \) is false.

For example, \([1+1=2] = 1\), \([1+1=1] = 0\), \([\text{it's currently snowing}] = 0\).

**Proof of Prop. 1.20:** Three cases, as in Thm 1.13.

**Proof of Thm 1.14:** We want to use induction, but we don’t have an \( \mathbb{N} \)-variable. So let’s first prove the “\( n \in \mathbb{N} \)” case:

**Observation 1:** \( \binom{n}{k} \in \mathbb{N} \) whenever \( n \in \mathbb{N}, \ k \in \mathbb{Z} \).

**Proof of Observation 1:** Induction on \( n \).

**Base case:** \( \binom{0}{k} = [k=0] \in \{0,1\} \subseteq \mathbb{N} \).
Step: Let \( m \in \mathbb{N} \). Assume (as the IH = induction hypothesis) that Observation 1 holds for \( n = m \).

We must prove that Observation 1 holds for \( n = m + 1 \).

Let \( k \in \mathbb{Z} \). Then, Thm. 1.12 says

\[
\binom{m+1}{k} = \binom{m+1-1}{k-1} + \binom{m+1-1}{k}
\]

\[
= \frac{m}{k-1} + \frac{m}{k} \quad \in \mathbb{N},
\]

(by IH) (by IH)

Thus, Observation 1 holds for \( n = m + 1 \).

So, Observation 1 is proven.

Now, prove Thm. 1.14:

**CASE 1:** \( n \geq 0 \).

**CASE 2:** \( n < 0 \).
In Case 1: \( n \in \mathbb{N} \), so Observation 1 yields 
\[
\binom{n}{k} \in \mathbb{N} \subseteq \mathbb{Z}.
\]

In Case 2: \( n < 0 \). So \( n \leq -1 \).
Prop 1.11b (applied to \(-n\) instead of \(n\)) yields
\[
\binom{n}{k} = (-1)^k \binom{-n+k-1}{k}.
\]
Thus, if \(-n+k-1 \in \mathbb{N}\), then Observation 1 yields
\[
\binom{n}{k} = (-1)^k \binom{-n+k-1}{k} \in \mathbb{Z},
\]
everywhere.

What if \(-n+k-1 \notin \mathbb{N}\)? Then, \(-n+k-1 < 0\).
Add \( n \leq -1 \), obtain \( k-1 \leq -1 \), so \( k < 0 \),
so \( \binom{n}{k} = 0 \in \mathbb{Z} \).
\( \square \)
Proof of Thm. 1.15, Induction on n.

Base: If S is a 0-elt. set, then
\[
\# \text{ of } k\text{-elt. subsets of } S \\
= \# \text{ of } k\text{-elt. subsets of } \emptyset \\
= \sum_{k=0}^{k} = \binom{0}{k} \quad \text{(by Prop. 1.20)}.
\]

Step: Let m \in \mathbb{N}, Assume (as IH) that Thm. 1.15 holds for n = m+1.

Assume (as IH) that Thm. 1.15 holds for n = m+1.

Let S be an (m+1)-elt. set.

Let S be an (m+1)-elt. set.

Thus, S is nonempty. So \exists t \in S. Fix such a t.

Thus, S is nonempty. So \exists t \in S. Fix such a t.

Now, there are two types of subsets of S:

1. Type 1: those that contain t.
2. Type 2: those that don't contain t.
So

\[
\# \text{ of } k\text{-elt. subsets of } S
= (\# \text{ of } k\text{-elt. subsets of } S \text{ of Type 1})
+ (\# \text{ of } k\text{-elt. subsets of } S \text{ of Type 2}).
\]

\[= (\# \text{ of } k\text{-elt. subsets of } S) - \# \text{ of the subsets of } S \text{ of Type 2}
= (\binom{m}{k})
= (\binom{m}{k}).
\]

But the subsets of $S$ of Type 2 are the subsets of $S \setminus \{t\}$. So

\[
(\# \text{ of } k\text{-elt. subsets of } S \text{ of Type 2})
= (\# \text{ of } k\text{-elt. subsets of } S \setminus \{t\}) = (\binom{m}{k}).
\]

What about Type 1?

Informally: The subsets of $S$ of Type 1 with $k$ elements
"correspond to" the subsets of $S \setminus \{t\}$ with $k-1$ elements.

Formally: The map
\( \{ k \text{-elt. subsets of } S \text{ of Type } 1 \} \rightarrow \{ (k-1) \text{-elt. subsets of } S \setminus \{t\} \} \)

is a bijection (its inverse is
\( \{ (k-1) \text{-elt. subsets of } S \setminus \{t\} \} \rightarrow \{ k \text{-elt. subsets of } S \text{ of Type } 1 \} \)

\( R \rightarrow R \cup \{t\} \).

But if \( X \) and \( Y \) are finite sets and if \( \exists \) bijection from
\( X \) to \( Y \), then \( |X| = |Y| \).

Thus, \( |\{ k \text{-elt. subsets of } S \text{ of Type } 1 \}| = |\{ (k-1) \text{-elt. subsets of } S \setminus \{t\} \}| \).

In other words,
\[
\binom{m}{k} = \binom{m}{k-1} \cdot \binom{m-1}{k-1} \text{ IH } (k-1).
\]
So (4) becomes

\[ \# \text{ of } k\text{-ett. subsets of } S \]

\[ = \binom{m}{k-1} + \binom{m}{k} \]

\[ \text{Thm 1.12 (applied to } n=m+1) \]

\[ \binom{m+1}{k} \]

This completes the step. \( \square \)

Thus, Thm 1.15 holds for \( n=m+1 \).

---

Proof of Thm. 1.16,

I will rewrite

\[ \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \]

as

\[ \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k} \]

(that is, a sum over all integers \( k \)).

Why is this infinite sum well-defined?

Because only finitely many of its addends are \( \neq 0 \).

For example, if \( n=3 \), it has the form

\[ \cdots + 0 + 0 + 0 + x^3 + 3x^2 y + 3xy^2 + y^3 + 0 + 0 + 0 + \cdots \]
An infinite sum that has only finitely many nonzero terms is always well-defined. Its value is obtained by discarding the 0's.

Since \( \binom{n}{k} = 0 \) whenever \( k \notin \{0, 1, \ldots, n\} \), we see that \[ \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \] is well-defined & equals \[ \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \].

So it remains to prove

\[ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \].

Now we'll prove this by induction:

**Base:** easy.

**Step:** Assume (2) holds for \( n=m \) (where \( m \in \mathbb{N} \) & fixed),

Must prove (2) holds for \( n=m+1 \).
We have

\[(x+y)^{m+1} = (x+y)^m (x+y)\]

\[= \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m-k} (x+y)\]

(by IH)

\[= \left( \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m-k} \right) (x+y)\]

\[= \left( \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m-k} \right) x + \left( \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m-k} \right) y\]

\[= \sum_{k \in \mathbb{Z}} \binom{m}{k} x^{k+1} y^{m-k} + \sum_{k \in \mathbb{Z}} \binom{m}{k} x^k y^{m-k+1}\]
\[ \sum_{k=2}^{m} \binom{m}{k-1} x^{(k-1)+1} y^{m-(k-1)} + \sum_{k=2}^{m} \binom{m}{k} x^{k} y^{m-k+1} \]

(here, we substituted \( k-1 \) for \( k \) in the 1st sum)

\[ \sum_{k=2}^{m} \binom{m}{k-1} x^{k} y^{m+1-k} + \sum_{k=2}^{m} \binom{m}{k} x^{k} y^{m+1-k} \]

\[ \sum_{k=2}^{m} \left( \binom{m}{k-1} + \binom{m}{k} \right) x^{k} y^{m+1-k} \]

\[ \sum_{k=2}^{m} \binom{m+1}{k} x^{k} y^{m+1-k} \]

\[ \sum_{k=2}^{m+1} \binom{m+1}{k} x^{k} y^{m+1-k} \]
Thus, (2) holds for $n = m + 1$.

(See [detnates, 32.2.1] for a proof without “infinite sums”.)

The proof of Thm. 1.16 is an exercise in induction.
The proof of Prop. 1.17 is an exercise on HW 1.

4. Counting

Counting := enumeration := finding sizes of finite sets.

E.g., what we have done in Thm. 4.6 & Thm. 4.15.

Much more can be done; some examples:

Def. A permutation of a set $X$ is a bijection $X \to X$.

For example, is a permutation of $\{0, 1, 5\}$. 

\[
\begin{align*}
0 &\rightarrow 0 \\
1 &\rightarrow 1 \\
5 &\rightarrow 5
\end{align*}
\]
Thm 1.21. Let n \in \mathbb{N}, let X be an n-elt. set.

\text{Then,} \quad \text{(\# of permutations of } X) = n!.

(Proof will be given later.)

Def. A derangement of a set X means a permutation \sigma of X such that \sigma(x) \neq x \ \forall x \in X.

How many derangements does an n-elt. set have?

Def. For each n \in \mathbb{N}, let \{n\} be the set \{1, 2, \ldots, n\}.

Instead of studying an arbitrary n-elt. set X, it suffices to study \{n\}:

Lem. 1.22. Let X be any n-elt. set. Then,

\[
(\# \text{ of derangements of } X) = (\# \text{ of derangements of } \{n\}).
\]

Proof. Fix a bijection \phi: X \rightarrow \{n\}. (It exists, since X has n elts.) Now, any derangement of X can be
transformed into a derangement of $[n]$ by "relabeling" the elements of $X$ as $1, 2, \ldots, n$. Using $\phi$.

For example, the derangement

\[
\begin{array}{ccc}
x & \omega & y \\
y & \omega & z \\
z & \omega & x
\end{array}
\]

becomes

\[
\begin{array}{ccc}
\phi^{-1} & \omega & \phi \\
1 & \omega & 2 \\
2 & \omega & 3 \\
3 & \omega & 1
\end{array}
\]

So, formally,

\[
\{\text{derangements of } X\} \rightarrow \{\text{derangements of } [n]\},
\]

\[
\omega \mapsto \phi \circ \omega \circ \phi^{-1}
\]

is a bijection (its inverse being
\{ \text{derangements of } [n] \} \rightarrow \{ \text{derangements of } X \}
\alpha \mapsto \phi^{-1} \circ \alpha \circ \phi.

Thus, \#(\text{derangements of } X) = \#(\text{derangements of } [n]), \quad \Box

So it suffices to count derangements of \([n]\).

**Def.** For each \(n \in \mathbb{N}\), let \(D_n = \#(\text{derangements of } [n])\).

**Def.** Let \(n \in \mathbb{N}\). The \underline{one-line notation} of a permutation \(\alpha\) of \([n]\) is the \(n\)-tuple \((\alpha(1), \alpha(2), \ldots, \alpha(n))\).

**Ex.** The permutations of \([3]\) are

\((1,2,3), (1,3,2), (2,1,3), (2,3,1)\),

\((3,1,2), (3,2,1)\).

Only \((2,3,1)\) and \((3,1,2)\) are derangements of \([3]\).

Thus \(D_3 = 2\).
Ex. \[ D_0 = 1 \quad \text{(since } id: \emptyset \to \emptyset \text{ is a derangement),} \]
\[ D_1 = 0 \quad \text{(since } id: [1] \to [1] \text{ is not a derangement),} \]
\[ D_2 = 1, \]
\[ D_3 = 2, \]
\[ D_4 = 9. \]

Thm. 1.20. (a) \[ |D_n| = (n-1)(D_{n-1} + D_{n-2}) \quad \forall n \geq 2. \]
(b) \[ D_n = n D_{n-1} + (-1)^n \quad \forall n \geq 1. \]
(c) \[ n! = \sum_{k=0}^{n} (n \choose k) D_k \quad \forall n \in \mathbb{N}. \]
(d) \[ D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \]
(e) \[ D_n = \text{round}\left(\frac{n!}{e}\right) \quad \forall n \geq 1, \quad \text{(where } e \approx 2.71828\ldots) \]
\[ = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor \]