4. THE TWELVEFOLD WAY

4.4. WHAT IS IT?

The twelvefold way is a table of $4 \times 3 = 12$ standard counting problems that frequently appear.

Informal description: Given a set $A$ of balls, and a set $X$ of boxes.
A placement means a way to distribute the balls into the boxes.

Rigorously: a placement is a map from $A$ to $X$.
At least, this is what will be called the "$L \rightarrow L$ placements."

How many placements are there? $|X|^{|A|}$.

Example: $|X| = 2, |A| = 3$.
For example, take $X = \{1, 2\}$ and $A = \{1, 2, 3\}.$
Always draw boxes in increasing order: \[
\begin{array}{llll}
1 & 2 & 3
\end{array}
\]
Here are the 8 $L \rightarrow L$ placements:
The order of the balls in a single box doesn't matter:

\[
\begin{array}{c}
\boxed{\boxed{1 2 3}} \\
\boxed{1 3} \\
\boxed{2 3} \\
\boxed{1 2 3}
\end{array}
\quad = 
\begin{array}{c}
\boxed{\boxed{1 3 2}} \\
\boxed{3} \\
\boxed{2 3} \\
\boxed{1 2 3}
\end{array}
\]

This suggests the following variations:
- What if we require \( f : A \rightarrow X \) to be injective (i.e., each box contains \( \leq 1 \) ball), or surjective (i.e., each box contains \( \geq 1 \) ball)?
What if the balls are unlabelled (i.e., indistinguishable)?

Note that we have not made this rigorous, but the gist is that we treat 
\[
\begin{array}{c}
1 \\
2 \ 3
\end{array}
\quad \text{and} \quad
\begin{array}{c}
2 \\
1 \ 3
\end{array}
\]
as the same placement.
We will see how to make this rigorous.

What if the boxes are unlabelled? i.e., what if we treat 
\[
\begin{array}{c}
1 \\
2 \ 3
\end{array}
\quad \text{and} \quad
\begin{array}{c}
2 \ 3 \\
1
\end{array}
\]
as the same placement?

What if both balls and boxes are unlabelled?

So we get \( 3 \cdot 4 = 12 \) different counting problems.

List them as a table:
<table>
<thead>
<tr>
<th>$A \rightarrow X$</th>
<th>arbitrary</th>
<th>injective</th>
<th>surjective</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L \rightarrow L$</td>
<td>$</td>
<td>X</td>
<td>^{</td>
</tr>
<tr>
<td>$U \rightarrow L$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$L \rightarrow U$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$U \rightarrow U$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For example, $L \rightarrow U$ means "balls are labelled, boxes are unlabelled".

The goals of this chapter are:
- Formulate $U \rightarrow L$, $L \rightarrow U$, and $U \rightarrow U$.
- Fill in the remaining 11 cells.
- See some examples.
Example: \(|X| = 2, \quad |A| = 3\).

|        | arbitrary | injective \((\text{since} \, |x| < |A|)\) | surjective |
|--------|-----------|--------------------------------|------------|
| \(L \rightarrow L\) | 8         | 0                               | 6          |
| \(u \rightarrow L\) | 4         | 0                               | 2          |
| \(L \rightarrow u\) | 4         | 0                               | 3          |
| \(u \rightarrow u\) | 2         | 0                               | 1          |

(e.g.) arbitrary \(L \rightarrow u\) placements:

\[
\begin{array}{c}
\text{①, ②, ③, ④, ⑤} \\
\text{①, ②, ③}
\end{array}
\]

In general: Not each of the 12 questions has a closed-form solution. But there are good recursive answers.
L→L placements are just maps A→X.

Prop. 4.1. \( (# \text{ of } L\rightarrow L \text{ placements } A \rightarrow X) = |X|^{|A|} \).

Proof. This is Theorem 3.4. \( \square \)

Prop. 4.2. \( (# \text{ of injective } L\rightarrow L \text{ placements } A \rightarrow X) = (# \text{ of injective maps } A \rightarrow X) \)

\[ = |X| (|X|-1) (|X|-2) \cdots (|X|-|A|+1), \]

Proof. This is Theorem 3.5. \( \square \)

Prop. 4.3. \( (# \text{ of surjective } L\rightarrow L \text{ placements } A \rightarrow X) = (# \text{ of surjective maps } A \rightarrow X) \)

\[ = \text{sur}(|A|, |X|), \]

Proof. This is Proposition 3.9. \( \square \)

(See HW3 exercise 2 for a formula for \( \text{sur}(n,k) \).)
Typical applications of $L \to L$ placements:
- assigning grades (from a finite set $X$) to students (from a finite set $A$);
- $L \to L$ placements (arbitrary);
- assigning IP addresses to a bunch of computers;
- injective $L \to L$ placements.

- How many 8-digit telephone numbers are there with no 2 equal digits?

Injective $L \to L$ placements (with $A = \{8\}$ and $X = \{0, 1, \ldots, 9\}$)

$\begin{array}{cccccc}
\underline{2} & \underline{4} & \underline{3} & \underline{5} & \underline{2} & \underline{6} \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}$

$\Rightarrow$ telephone number 203549686

$\Rightarrow$ the number of such numbers is $10 \cdot 9 \cdots 3 = \frac{10!}{2!}$. 
Remark: Here's a quick problem NOT from the twelfold way:

How many 8-digit telephone numbers contain no 2 adjacent equal digits?

(e.g. 31315315 is okay, but 12334567 isn't.)

Answer:

\[
\begin{align*}
10 & \quad \uparrow \\
\text{options for 1st digit} & \quad \uparrow \\
9 & \quad \uparrow \\
\text{options for 2nd digit} & \quad \uparrow \\
9 & \quad \uparrow \\
\text{options for 3rd digit} & \quad \ldots \\
9 & \\
\end{align*}
\]

\[= 10 \cdot 9^7.\]

4.3. UNLABELLED OBJECTS

What does it mean for balls, or boxes, to be unlabelled?

The idea is that (with unlabelled boxes) we want to treat \[\begin{array}{c}
\{1, 2, 3\}
\end{array}\] and \[\begin{array}{c}
\{2, 3, 4\}
\end{array}\] as the
The rigorous way to do this is by introducing an equivalence relation & passing to equivalence classes.

(References for equivalence classes: notes/handouts by Melissa Lynn, as linked from the class website.)

Def. A (binary) relation on a set $S$ is a subset of $S \times S$.

Idea: something like $=$ or $\leq$ or $\leq$ or $|$ (divides), or $|$ (divisible by), or $\equiv$, or $\neq$, or $\sim$ (many more).

"$\equiv \text{mod } k$" for a given $k \in \mathbb{Z}$, or $\sim$ for geometric shapes, .... (many more).

If $R$ is a relation on $S$, then we write $a R b$ if and only if $(a, b) \in R$.

For example, the relation $\leq$ on $\mathbb{N}$ is really the set of all $(a, b) \in \mathbb{N} \times \mathbb{N}$ with $a \leq b$. 

Def. An equivalence relation on a set $S$ is a relation $\sim$ which is:

- reflexive. (i.e., it satisfies $a \sim a \ \forall a \in S$);
- symmetric (i.e., if $a \sim b$, then $b \sim a$);
- transitive (i.e., if $a \sim b$ and $b \sim c$, then $a \sim c$).

Idea: An equivalence relation relates objects that we want to treat as equal.

Examples: $=$ is an equivalence relation (on any set).

"$\equiv \mod k$" is an equivalence relation $\forall k \in \mathbb{Z}$,

$\leq$ is not an equiv. rel. (it is reflexive & transitive, but not symmetric).

$\neq$ is not (it is symmetric but neither reflexive nor transitive).

$\sim$ for geometric shapes is an equiv. rel.,

$\parallel$ for lines in the plane.
Now, go back to balls & boxes:

Def. Let $f, g : A \to X$. Then we say that

- $f$ is box-equivalent to $g$ (written $f \boxsim g$) if & only if $\exists$ permutation $\sigma$ of $X$ such that $f = \sigma \circ g$ (in other words, $f$ can be obtained from $g$ by permuting boxes).

- $f$ is ball-equivalent to $g$ (written $f \ballsim g$) if & only if $\exists$ permutation $\tau$ of $\pi(A)$ such that $f = g \circ \tau$ (in other words, $f$ can be obtained from $g$ by permuting balls).

- $f$ is box-ball-equivalent to $g$ (written $f \boxballsim g$) if & only if $\exists$ permutation $\sigma$ of $X$ & $\tau$ a permutation $\tau$ of $A$ such that $f = \sigma \circ g \circ \tau$. 

Examples:

\[
\begin{array}{c}
\text{box} \quad \sim \\
\text{box} \\
\text{ball}
\end{array}
\quad \begin{array}{c}
\text{box} \\
\text{ball}
\end{array}
\]

All of \( \sim \), \( \sim \), and \( \sim \) are equivalence relations. So counting \( U \rightarrow L \) placements should mean treating ball-equivalent labelings as identical. How to do that?

- Count equivalence classes.

**Def.** Let \( \sim \) be an equivalence relation on a set \( S \).

- Let \( x \in S \). Then, the \( \sim \)-equivalence class of \( x \), denoted by \([x]_\sim\), is defined by

\[
[x]_\sim = \{ y \in S \mid y \sim x \}.
\]
Examples: in \( \mathbb{N} \), we have

\[
[5] = \{ y \in \mathbb{N} \mid y \equiv 5 \pmod{5} \} = \{5, 10, 15, \ldots \}.
\]

\[
[5] \equiv_{\pmod{3}} = \{ y \in \mathbb{N} \mid y \equiv 5 \pmod{3} \} = \{2, 5, 8, 11, 14, \ldots \}.
\]

\[
[5] \equiv_{\pmod{2}} = \{1, 3, 5, 7, \ldots \}.
\]

In our running example with \(|X| = 2\) and \(|A| = 3\), we have

\[
\begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix}.
\]
Crucial fact about equivalence classes:

Let $\sim$ be an equivalence relation on a set $S$.

Let $x \in S$ and $y \in S$.

Then $x \sim y \iff [x]_{\sim} = [y]_{\sim}$.

Thus, "counting elements of $S$ up to $\sim$-equivalence"

$=$ counting distinct $\sim$-equivalence classes.
Def. A \( U \rightarrow L \) placement (i.e., placement of Unlabelled balls into Labelled boxes) is a \( \sigma \)-equivalence equivalence class of maps \( f: A \rightarrow X \).

Example. For \( |X| = 2 \) and \( |A| = 3 \), here are the \( U \rightarrow L \) placements (drawn as circles):
Notation: When visualizing a $U \rightarrow L$ placements, we just draw the balls as circles, with no numbers in them. So the 4 $U \rightarrow L$ placements for $|X| = 2$ and $|A| = 3$ are

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
1 \\
1 \\
1 \\
1 \\
\end{array}
\end{array}
\]

Prop. 4.5, \( \# \text{ of } U \rightarrow L \text{ placements } A \rightarrow X \) \( = (\# \text{ of } (x_1, \ldots, x_{|X|}) \in \mathbb{N}^{\times|X|} \text{ satisfying } x_1 + \ldots + x_{|X|} = |A|) \)

\[
= \binom{|A| + |X| - 1}{|A|}.
\]

Proof. 1st equality: Consider the bijection \( \{ U \rightarrow L \text{ placements} \} \rightarrow \{ \text{weak compositions of } |A| \text{ into } |X| \text{ parts} \} \),

\[
\begin{array}{c}
\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array}
\end{array}
\]

(Fine print: We need to assume that $X = [1, \ldots, |X|]$.)
(We are not doing the rigorous argument.)

Thus,

\( \text{(\# of } u \rightarrow l \text{ placements)} \)

\( = (\# \text{ of weak compositions of } |A| \text{ into } |x| \text{ parts}) \)

\( = (\# \text{ of } (x_1, \ldots, x_{|x|}) \in \mathbb{N}^{|x|} \text{ satisfying } x_1 + \cdots + x_{|x|} = |A|) \). \quad \Box

2nd equality: Theorem 3.25.

Prop. 4.6. \( (\# \text{ of surjective } u \rightarrow l \text{ placements}) \)

\( = (\# \text{ of } (x_1, \ldots, x_{|x|}) \in \{1, 2, 3, \ldots\}^{|x|} \text{ satisfying } x_1 + \cdots + x_{|x|} = |A|) \)

\[ = \begin{cases} \binom{|A| - 1}{|x| - 1} & \text{if } |A| \geq |x|; \\ \lfloor |x| = 0 \rfloor & \text{if } |A| = 0 \end{cases} \]

\[ = \binom{|A| - 1}{|A| - |x|}. \]
Proof. 1st equality: same argument as in Prop. 4.5.

2nd equality: Theorem 3.23.

3rd equality: symmetry of Pascal's triangle.

\[ \text{Prop. 4.7, (} \# \text{ of injective } U \to L \text{ placements)} \]

\[ = (\# \text{ of } (x_1, \ldots, x_{|x|}) \in \{0,1\}^{|x|} \text{ satisfying } x_1 + \ldots + x_{|x|} = |A|) \]

\[ = \binom{|x|}{|A|}. \]

\[ \text{Proof. 1st equality: same argument as in Prop. 4.5.} \]

\[ \text{2nd equality: Theorem 3.24.} \]

**4.5. L \to U**

**Def.** An \( L \to U \) placement is a box-equivalence class of maps \( f: A \to X \).
Example: \( |X| = 2, \ |A| = 3 \).
21 boxes equivalent identically:

E.g.

\[
\begin{array}{cccc}
1 & 2 & 1 & 4 \\
& & 3 &
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
& & &
\end{array}
\]

(since they each consist of 1 box with "ball 1", 1 box with "ball 2", ..., 1 box with "ball |A|", and |X| - |A| empty boxes),

so the # of equivalence classes is 1.

Prop. 4.9.

\[
\# \text{ of surjective } L \to U \text{ placements}
\]

\[
= \left\{ \begin{array}{l}
|A| \\
|X| \end{array} \right\}
\]

(2 Stirling number of the 2nd kind, as defined on HW3 by \( \frac{s_{n}}{k!} = \frac{\text{sur}(n,k)}{k!} \))

\[
= \frac{\text{sur}(|A|,|X|)}{|X|!}
\]
Proof. With $X = \{1, 2\}$,

Then, the surjective $L \to U$ placements are in bijection with the set partitions of $A$ into $1 \times 1$ parts.

\[
\begin{bmatrix}
1 & 3 & 5 \\
35 & 2 & 4
\end{bmatrix}
\]

\[
\text{Box}
\]

\[
= \{ \{1\}, \{3, 5\} \}
\]

So
\[
\text{(\# of surjective $L \to U$ placements)}
\]

\[
= (\text{\# of set partitions of } A \text{ into } 1 \times 1 \text{ parts})
\]

\[
= \sum_{A \subseteq \{1, 2\}} \binom{|A|}{1 \times 1}
\]

(by some remark in HW3)

\[
= \frac{\text{sur } (1|A|, 1|X|)}{|X|!}
\]

\[
\square
\]
Prop. 4.10, \((\#\) of \(L \to U\) placements) \(-22-\)

\[= \sum 1^1_j + \sum 1^2_j + \sum 1^3_j + \ldots + \sum 1^n_j.\]

Proof. Exercise. \(\Box\)