5. PERMUTATIONS

We will go deeper into this subject now, but not too deep. See [detnotes, Ch. 4] for details, and [Bóna: Combinatorics of Permutations] for much more.

Recall: a permutation of a set $X$ is a bijection from $X$ to $X$.

5.1. DEFINITIONS

Def. Given $n \in \mathbb{N}$, let $S_n$ denote the set of all permutations of $[n]$. This set $S_n$ is called the $n$-th symmetric group. It is closed under composition (i.e., if $\alpha \in S_n$ and $\beta \in S_n$, then $\alpha \circ \beta \in S_n$) and under inverses (i.e., if $\alpha \in S_n$, then $\alpha^{-1} \in S_n$).

Def. Let $n \in \mathbb{N}$ and $\alpha \in S_n$. Then, introduce 2 notations for $\alpha$:

(2) The one-line notation of $\alpha$ is the $n$-tuple $[\alpha(1), \alpha(2), \ldots, \alpha(n)]$. (The use of square brackets here is a standard.)
Often, one omits the commas & the brackets. E.g., the permutation

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 2 & 3 & 4 & 5
\end{pmatrix} \in S_5
\]

has the one-line notation \([4, 2, 1, 3, 5]\), or, short, \(42135\).

(b) The cycle digraph of \(\sigma\) is defined (informally) as follows:
For each \(i \in [n]\), draw 2 points ("node") labelled \(i\).
For each \(i \in [n]\), draw an arrow ("arc") from the node labelled \(i\) to the node labelled \(\sigma(i)\).
The result is called the cycle digraph of \(\sigma\).

E.g., the above permutation has cycle digraph

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5
\end{pmatrix}
\]
E.g., the permutation in $S_8$ whose one-line notation is $[8, 1, 4, 7, 5, 3, 6, 2]$ has a cycle digraph

Prop. 5.1. Let $n \in \mathbb{N}$. Then,

$s_n \rightarrow \{n\text{-tuples of distinct elements of } [n]\}$

$\circ \rightarrow \text{(one-line notation of } \circ) = [\circ(1), \circ(2), \ldots, \circ(n)]$

is a bijection.

Proof. Permutations of $[n]$ are the same as injective maps $[n] \rightarrow [n]$, by the Pigeonhole Principle for injections. $\square$
5.2. INVERSIONS & LENGTHS

Def. (2) Let \( i \) and \( j \) be two distinct elements of a set \( X \). Then, the transposition \( t_{i,j} \) is the permutation of \( X \) that sends \( i \) to \( j \), \( j \) to \( i \), and leaves everything else in its place.

If \( X = \{1, 2, \ldots, n\} \) for some \( n \in \mathbb{N} \), then the one-line notation of \( t_{i,j} \) is \( [1, 2, \ldots, i-1, j, i+1, \ldots, j-1, i, j+1, \ldots, n] \) if \( i < j \).

The cycle digraph of \( t_{i,j} \) is

\[ \circlearrowleft ! \circlearrowleft ! \circlearrowleft ! \circlearrowleft ! \circlearrowleft ! \circlearrowleft \]

(b) Let \( n \in \mathbb{N} \) and \( i \in \{1, 2, \ldots, n-1\} \). Then the simple transposition \( s_i \in S_n \) is defined by \( s_i = t_{i, i+1} \).

(This was used in Midterm 1 Exercise 2.)
Prop. 5.2. Let $n \in \mathbb{N}$.

(a) $s_i^2 = \text{id}$ for all $i \in [n-1]$.

(Recall $s_i^2 = s_i \circ s_i$.)

(b) $s_i \circ s_j = s_j \circ s_i$ for all $i, j \in [n-1]$ with $|i-j| > 1$.

(c) $s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1} = t_{i+1, i+2}$ for all $i \in [n-1]$. 

Proof. Everything follows from straightforward casework.

For (c), the following picture helps visualize the argument:

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & \ldots & i & i+1 & i+2 & \ldots & n \\
\end{array}
\]

\[
\text{Diagram:}
\]
so \( s_i \circ s_{i+1} \circ s_i = 	au_{i,i+2} \).
Def. Let \( n \in \mathbb{N} \), let \( \sigma_0 \) be the permutation in \( S_n \) sending each \( i \) to \( n+1-i \).

In other words, it "reflects" numbers across the middle of \([n]\). It is the unique strictly decreasing permutation of \([n]\).

Example: If \( n=5 \), then \( \sigma_0 = [5, 4, 3, 2, 1] \) in one-line notation, with cycle digraph \( 1 \xrightarrow{} 5 \xrightarrow{} 2 \xrightarrow{} 4 \xrightarrow{} 3 \xrightarrow{} \).

If \( n=6 \), then \( \sigma_0 = [6, 5, 4, 3, 2, 1] \) in one-line notation, with cycle digraph \( 1 \xrightarrow{} 6 \xrightarrow{} 2 \xrightarrow{} 5 \xrightarrow{} 3 \xrightarrow{} 4 \xrightarrow{} \).

Remark: \( \sigma_0 \) and \( \tau_{ij} \) (for all \( i,j \)) are involutions (i.e., permutations \( \sigma^2 = \text{id} \)).

Def. Let \( n \in \mathbb{N} \), let \( i_1, i_2, \ldots, i_k \) be \( k \) distinct elements of \([n]\).

Then, \( \text{cyc}\{i_1, i_2, \ldots, i_k\} \) means the permutation in \( S_n \) that sends
The cycle digraph of $\text{cyc}_{3, 5, 7, 6} \in S_8$ is

![Diagram of cycle digraph](image)

Example: The cycle digraph of $\text{cyc}_{3, 5, 7, 6} \in S_8$ is

Remark: People normally write $(i_1, i_2, ..., i_k)$ for $\text{cyc}_{i_1, i_2, ..., i_k}$

Prop. 5.3. Let $n \in \mathbb{N}$.

(a) For any $k$ distinct elements $i_1, i_2, ..., i_k$ of $[n]$, we have

\[ \text{cyc}_{i_1, i_2, ..., i_k} = \text{cyc}_{i_2, i_3, ..., i_k, i_1} \]

(b) For any $i \in [n]$ and $k \in \mathbb{N}$ such that $i+k-1 \leq n$, we have

\[ \text{cyc}_{i, i+1, ..., i+k-1} = \text{cyc}_{i+1, i+2, ..., i+k-1, i} \]
(c) \( \omega_0 = s_1 (s_2 s_1) (s_3 s_2 s_1) \cdots (s_{n-1} s_{n-2} \cdots s_1) \)
\[ = (s_1 s_2 \cdots s_{n-1}) (s_2 s_3 \cdots s_{n-2}) \cdots (s_1 s_2) s_1 \]

Here & in the following, \( \alpha \beta \) means \( \alpha \circ \beta \) when \( \alpha, \beta \in S_n \).

(d) If \( 1 \leq i < j \leq n \), then
\[ t_{i,j} = s_i s_{i+1} \cdots s_{j-1} \cdots s_{i+1} s_i \]
\[ = s_{j-2} s_{j-1} \cdots s_{i} \cdots s_{j-2} s_{j-1} \]

(Both products have \( 2(j-i) - 1 \) factors.)

(e) If \( i_1, i_2, \ldots, i_k \) are \( k \) distinct elements of \([n]\), and if \( o \in S_n \), then \( o \circ \text{cyc}_{i_1, i_2, \ldots, i_k} \circ o^{-1} = \text{cyc}_{o(i_1), o(i_2), \ldots, o(i_k)} \).

(f) \( t_{i,j} = \text{cyc}_{i,j} \).

(g) \( s_i = \text{cyc}_{i,i+1} \).

(h) \( \text{cyc}_i = \text{id} \).
Proof. (a) Applying \( t_{i_1, i_2} \circ t_{i_2, i_3} \circ \ldots \circ t_{i_{k-1}, i_k} \) to \( i_j \) for some \( j \in [k-1] \), we get

\[
\begin{align*}
  i_j & \xrightarrow{t_{i_{k-1}, i_k}} i_j \xrightarrow{t_{i_k, i_{k-1}}} i_j \xrightarrow{t_{i_{j+1}, i_{j+1}}} i_j \xrightarrow{t_{i_{j+1}, i_{j+1}}} i_{j+1} \\
  & \xrightarrow{t_{i_2, i_3}} i_{j+1} \xrightarrow{t_{i_3, i_2}} i_{j+2} \xrightarrow{t_{i_3, i_2}} i_{j+2} \xrightarrow{t_{i_2, i_3}} i_{j+3} \xrightarrow{t_{i_2, i_3}} i_{j+3} \\
  & \ldots \xrightarrow{t_{i_2, i_3}} i_{j+1} \xrightarrow{t_{i_3, i_2}} i_{j+2} \xrightarrow{t_{i_2, i_3}} i_{j+3} \xrightarrow{t_{i_2, i_3}} i_{j+3}.
\end{align*}
\]

So

\[
(t_{i_1, i_2} \circ t_{i_2, i_3} \circ \ldots \circ t_{i_{k-1}, i_k})(i_j) = i_{j+1}
\]

(1)

Applying \( t_{i_1, i_2} \circ t_{i_2, i_3} \circ \ldots \circ t_{i_{k-1}, i_k} \) to \( i_k \), we get

\[
\begin{align*}
  i_k & \xrightarrow{t_{i_{k-1}, i_k}} i_{k-1} \xrightarrow{t_{i_{k-2}, i_{k-1}}} i_{k-2} \xrightarrow{t_{i_{k-2}, i_{k-1}}} i_{k-2} \xrightarrow{t_{i_{k-2}, i_{k-1}}} \ldots \xrightarrow{t_{i_2, i_3}} i_2 \xrightarrow{t_{i_2, i_3}} i_1.
\end{align*}
\]

So
\( (t_{i_{k-1}, i_{k}} \circ t_{i_{k-2}, i_{k-1}} \circ \cdots \circ t_{i_{1}, i_{2}})(i_k) = i_1 \)

\( = \text{cyc}_{i_{k-1}, i_{k-2}, \ldots, i_1, i_k}(i_k). \)

(2)

Finally, if \( x \in \text{cyc}_{i_1, i_2, \ldots, i_k} \), then

\( (t_{i_{k-1}, i_{k}} \circ t_{i_{k-2}, i_{k-1}} \circ \cdots \circ t_{i_1, i_2})(x) = x = \text{cyc}_{i_1, i_2, \ldots, i_k}(x). \)

Combining (1), (2) and (3), we see that

\( (t_{i_{k-1}, i_{k}} \circ t_{i_{k-2}, i_{k-1}} \circ \cdots \circ t_{i_1, i_2})(x) = \text{cyc}_{i_{k-1}, i_{k-2}, \ldots, i_1, i_k}(x) \quad \forall x \in \text{cyc}_{i_1, i_2, \ldots, i_k}. \)

Hence, \( t_{i_{k-1}, i_{k}} \circ t_{i_{k-2}, i_{k-1}} \circ \cdots \circ t_{i_1, i_2} = \text{cyc}_{i_{k-1}, i_{k-2}, \ldots, i_1, i_k}. \)

(b) Apply (a) to \( i_1 = i, \ i_2 = i+1, \ldots, i_k = i+k-1. \)

(c) For each \( k \in \{1, 2, \ldots, n-1\} \), we have \( s_1 s_2 \cdots s_k = \text{cyc}_{1, 2, \ldots, k+1} \)

(by part (b), applied to \( i = 1 \)).

Now, claim: For each \( k \in \{1, 2, \ldots, n-1\} \), the permutation

\( (s_{k+1} s_{k+2} \cdots s_{k-1}) (s_{k+2} s_{k+3} \cdots s_k) \cdots (s_{k+2} s_{k+3}) s_{k+1} \)

sends \( 1, 2, \ldots, k \) to \( k, k-1, \ldots, 1 \) but leaves \( k+1, k+2, \ldots, n \) in their places.
Induction over \( k \):

Induction base \( (k=1) \): (empty product of permutation) = \( \text{id} \).

Induction step \( (k \rightarrow k+1) \):

Assume the permutation

\[
\xi := (s_1 s_2 \ldots s_{k-1}) (s_1 s_2 \ldots s_{k-2}) \ldots (s_1 s_2) s_1
\]

sends \( 1, 2, \ldots, k \) to \( k, k-1, \ldots, 1 \) while leaving \( k+1, k+2, \ldots, n \) in their places. (Induction hypothesis.)

We must prove that the permutation

\[
\xi' := (s_1 s_2 \ldots s_k) (s_1 s_2 \ldots s_{k-1}) \ldots (s_1 s_2) s_1
\]

\[
= (s_1 s_2 \ldots s_k) \xi = \text{cyc } 1, 2, \ldots, k+1
\]

sends \( 1, 2, \ldots, k+1 \) to \( k+1, k, \ldots, 1 \) while leaving \( k+2, \ldots, n \) in their places.

In other words, we must prove

\[
(5) \quad \xi'(i) = k+2-i \quad \forall i \in \{k+1\}, \quad \text{and}
\]

\[
(6) \quad \xi'(i) = i \quad \forall i > k+1.
\]
Proof of (5): Let $i \in \mathbb{E}[k+1]$.

If $i \in \mathbb{E}[k]$, then the inductive hypothesis yields $\xi(i) = k+1 - i$, but (4) yields

$$\xi'(i) \overset{(4)}{=} \text{cyc}_{1, \ldots, k+1} \frac{\xi(i)}{= k+1 - i} = \text{cyc}_{1, \ldots, k+1} \frac{(k+1 - i)}{< k+1}$$

$$= (k+1 - i) + 1 = k+2 - i.$$

If $i = k+1$, then

$$\xi'(i) \overset{(4)}{=} \text{cyc}_{1, \ldots, k+1} \frac{\xi(i)}{= k+1} = \text{cyc}_{1, \ldots, k+1} \frac{(k+1)}{= \xi'(k+1) = k+1}$$

(by the inductive hypothesis)

$$= 1 = \frac{k+2 - (k+1)}{= 1} = k+2 - i.$$

So (5) is proven in both cases $i \in \mathbb{E}[k]$ and $i = k+1$.

Proof of (6): Let $k$.

This completes the induction step. $\implies$ The claim is proven.

Applying the claim to $k=n$, we conclude that the permutation $(s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1$ sends $1, 2, \ldots, n$
to \( n, n-1, \ldots, 1 \). Thus, it equals \( w_0 \), so

\[(7) \quad w_0 = (s_1 s_2 \cdots s_{n-2}) (s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1.\]

Next, we want to show

\[(8) \quad w_0 = s_1 (s_2 s_1) \cdots (s_{n-2} s_{n-3} \cdots s_1) (s_{n-1} s_{n-2} \cdots s_1).\]

This is proven similarly, or can be derived from (7) as follows:

Recall \((x_1 x_2 \cdots x_k)^{-1} = x_k^{-1} x_{k-1}^{-1} \cdots x_1^{-1}\) for any \(k\).

By inverting (7), we get

\[w_0^{-1} = \left((s_1 s_2 \cdots s_{n-1}) (s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2) s_1\right)^{-1}\]

\[= s_1^{-1} (s_2^{-1} s_1^{-1}) \cdots (s_{n-2}^{-1} s_{n-3}^{-1}) (s_{n-1}^{-1} s_{n-2}^{-1} s_{n-3}^{-1})\]

\[= s_1 (s_2 s_1) \cdots (s_{n-2} s_{n-3} \cdots s_1) (s_{n-1} \cdots s_2 s_1)\]

(since \(s_i^{-1} = s_i \quad \forall i\))

\[= s_1 (s_2 s_1) \cdots (s_{n-2} s_{n-3} \cdots s_1) (s_{n-1} s_{n-2} \cdots s_1)\]

\[w_0 \quad (\text{since } w_0 \text{ is an involution}).\]

So (8) is proven. This proves (c).
Def. Let \( n \in \mathbb{N} \) and \( \sigma \in S_n \).

1. An inversion of \( \sigma \) is a pair \((i,j)\) of elements of \( \mathbb{N} \) such that \( i < j \) and \( \sigma(i) > \sigma(j) \).

2. The length of \( \sigma \) is the number of inversions of \( \sigma \).

Example: Let \( \pi = [3,1,4,2] \in S_4 \).

The inversions of \( \pi \) are \((1,2)\) (since \( 1 < 2 \) and \( \pi(1) = 3 \neq \pi(2) = 2 \)), \((1,4)\) (since \( 1 < 4 \) and \( \pi(1) = 3 \neq \pi(4) = 1 \)), and \((3,4)\).

So the length of \( \pi \) is 3.

Def. The length of a permutation \( \sigma \in S_n \) is called \( l(\sigma) \).

Remark: If \( \sigma \in S_n \), then \( 0 \leq l(\sigma) \leq \binom{n}{2} \).
The only $\sigma \in S_n$ with $l(\sigma) = 0$ is $\text{id}$.

(Indeed, if $l(\sigma) = 0$, then every $\sigma$ has no inversions, so $\forall i < j$ satisfy $\sigma(i) \leq \sigma(j)$, so $\sigma(1) \leq \sigma(2) \leq \ldots \leq \sigma(n)$, so $\sigma(1) < \sigma(2) < \ldots < \sigma(n)$, so $\sigma = \text{id}$.)

The only $\sigma \in S_n$ with $l(\sigma) = (\binom{n}{2})$ is $\omega_0$.

In between, there are many:

\[ S_3 \]

(Permutation in one-line notation)

\[
\begin{align*}
321 & \quad \leftarrow \quad l=3 \\
231 & \quad \leftarrow \quad l=2 \\
132 & \quad \leftarrow \quad l=1 \\
123 & \quad \leftarrow \quad l=0
\end{align*}
\]

We draw an edge
\[ \alpha \xrightarrow{s_i} \beta \]
if $\alpha = \beta \circ s_i$
(or equivalently, $\beta = \alpha \circ s_i$).
The n-permutahedron

To each vertex \( (a_1, a_2, \ldots, a_n) \) in \( \mathbb{Z}_{n+1} \), assign the point \((0, a_1, a_2, \ldots, a_n)\) in \( \mathbb{R}^n \).

Then, these \( n! \) points are the vertices of the n-permutahedron.