Recall: An inversion of a permutation \( \sigma \in S_n \) is a pair 
\((i, j)\) satisfying \(1 \leq i < j \leq n\) and \(\sigma(i) > \sigma(j)\).

The length \( l(\sigma) \) of a permutation \( \sigma \in S_n \) is the \# of inversions of \( \sigma \).

Example: If \( \sigma = [3, 1, 5, 2, 4] \in S_5 \) (in one-line notation), then

- the inversions of \( \sigma \) are \((4, 2), (1, 4), (3, 4), (3, 5)\);
- the length of \( \sigma \) is 4.

Furthermore, for this \( \sigma \), we have \( \sigma \circ s_1 = [1, 3, 5, 2, 4] \), and

- the inversions of \( \sigma \circ s_1 \) are \((2, 4), (3, 4), (3, 5)\);
- the length of \( \sigma \circ s_1 \) is 3.

Also, for this \( \sigma \), we have \( \sigma \circ s_2 = [3, 5, 1, 2, 4] \), and

- the inversions of \( \sigma \circ s_2 \) are \((1, 3), (1, 4), (2, 3), (2, 4), (2, 5)\);
- the length of \( \sigma \circ s_2 \) is 5.

General rule: The one-line notation for \( \sigma \circ s_k \) (where \( k \in [n-1] \)) is obtained from the one-line notation for \( \sigma \) by swapping the \( k \)-th and \((k+1)\)-st entries. The effect on
Inversions is:

- If \( \sigma(k) < \sigma(k+1) \), then \( \sigma \circ s_k \) has a new inversion \((k, k+1)\).
  If \( \sigma(k) = \sigma(k+1) \), then \( \sigma \) had an inversion \((k, k+1)\)
  which \( \sigma \circ s_k \) no longer has.

- Any inversion \((i, j)\) of \( \sigma \) with \((i, j) \neq (k, k+1)\) gives rise to an inversion \((s_k(i), s_k(j))\) of \( \sigma \circ s_k \).

- This covers all inversions of \( \sigma \circ s_k \).

\[\Rightarrow\]

\[ l(\sigma \circ s_k) = l(\sigma) + 1 \text{ if } \sigma(k) < \sigma(k+1); \]
\[ l(\sigma \circ s_k) = l(\sigma) - 1 \text{ if } \sigma(k) > \sigma(k+1). \]

Prop. 5.4. For any \( \sigma \in S_n \), we have \( l(\sigma^{-1}) = l(\sigma) \).

Proof. There is a bijection

\[ \{ \text{inversions of } \sigma \} \leftrightarrow \{ \text{inversions of } \sigma^{-1} \}, \]

\[ (i, j) \leftrightarrow (\sigma^{-1}(j), \sigma^{-1}(i)) \]

(Its inverse map sends \((i, j)\) to \((\sigma(i), \sigma(j))\).)

For details: [lecture notes, Exercise 4.2 (f)]. \( \square \)
Prop. 5.5 Let $n \in \mathbb{N}$, $\sigma \in S_n$ and $k \in \{1, \ldots, n-1\}$.

(a) We have

$$l(\sigma s_k) = \begin{cases} l(\sigma) + 1, & \text{if } \sigma(k) < \sigma(k+1); \\ l(\sigma) - 1, & \text{if } \sigma(k) > \sigma(k+1). \end{cases}$$

(b) We have

$$l(\sigma s_k \circ \sigma^{-1}) = \begin{cases} l(\sigma) + 1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ l(\sigma) - 1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

(Note: $\sigma^{-1}(i)$ is the position in which the entry $i$ appears in the one-line notation of $\sigma$.)

Proof. (2) was essentially shown in the Example above — see (1).

(b) Applying part (a) to $\sigma^{-1}$ instead of $\sigma$, we get

$$l(\sigma^{-1} s_k) = \begin{cases} l(\sigma^{-1}) + 1, & \text{if } \sigma^{-1}(k) < \sigma^{-1}(k+1); \\ l(\sigma^{-1}) - 1, & \text{if } \sigma^{-1}(k) > \sigma^{-1}(k+1). \end{cases}$$

But Prop. 5.4 yields $l(\sigma^{-1}) = l(\sigma)$.

Also, $l(\sigma^{-1} s_k) = l((s_k \circ \sigma^{-1})^{-1}) = l(s_k^{-1} \circ \sigma^{-1})$ (Prop. 5.4)}
Thus, (2) transforms into the equality we're proving.

(For details, see [deBroske's Exercise 4.2(a)].) 0

Theorem 5.6.

Let \( n \in \mathbb{N} \), let \( \sigma \in S_n \). Then, \( \ell(\sigma) \) is the minimum \( p \in \mathbb{N} \) such that \( \sigma \) can be written as a composition of \( p \) simple transpositions (i.e., transpositions of the form \( s_k \)).

[Keep in mind: The composition of 0 transpositions is \( \text{id.} \).]

Example: In \( S_4 \), we have

\[
[4, 1, 3, 2] = s_2 s_3 s_2 s_4 = \underbrace{s_3 s_2 s_3}_{3} s_1 s_3 = s_3 s_2 s_1 s_3 = \ldots
\]

Proof of Thm. 5.6.

Claim 1:

\( \sigma \) can be written as a composition of \( \ell(\sigma) \) simples (= simple transpositions).
Claim 2: $\sigma$ cannot be written as a composition of $<l(\sigma)$ simples.

Proof of Claim 1: Induction on $l(\sigma)$.

**Base:** If $l(\sigma) = 0$, then $\sigma = id$, so $\sigma$ is a composition of 0 simples.

**Step:** Assume (as the IH) that Claim 1 holds for $l(\sigma) = h$. Now, let $\sigma \in S_n$ be such that $l(\sigma) = h+1$. Then, $\sigma \neq id$ (since $l(\sigma) = h+1 > 0$).

Hence, $\exists k \in [n-1]$ such that $\sigma(k) > \sigma(k+1)$.

Fix such a $\sigma$. Then, Prop. 5.5 (2) yields

$l(\sigma \circ s_k) = l(\sigma) - 1 = h+1 - 1 = h$.

Hence, the IH (applied to $\sigma \circ s_k$ instead of $\sigma$) yields that $\sigma \circ s_k$ can be written as a product of $h$ simples: $\sigma \circ s_k = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_h}$. 
Thus,
\[ \sigma = s_i \circ s_{i-1} \circ \ldots \circ s_{k-1} \overset{k}{=} s_k = s_i \circ s_{i-1} \circ \ldots \circ s_{k-1}. \]

This shows that \( \sigma \) is a composition of \( k+1 = l(\sigma) \) simples.

So Claim 1 holds for \( l(\sigma) = k+1 \).

This completes the inductive proof of Claim 1.

[Underlying idea of the above proof: bubblesort.]

Proof of Claim 2: Prop. 5.5 (2) yields
\[ l(\sigma \circ s_k) \leq l(\sigma) + 1 \quad \forall \sigma \in S_n \text{ and } k \in [n-1], \]
Thus, \( \forall k_1, k_2, \ldots, k_p \in [n-1], \) then
\[ l(s_{k_1} \circ s_{k_2} \circ \ldots \circ s_{k_p}) \leq l(s_{k_1} \circ s_{k_2} \circ \ldots \circ s_{k_{p-1}}) + 2 \]
\[ \leq l(s_{k_1} \circ s_{k_2} \circ \ldots \circ s_{k_{p-1}}) + 1 \]
\[ \leq \ldots \leq l(id) + p = p. \]
Now, if \( \sigma \) was the composition of \( p < \ell(\sigma) \) simples \( s_{k_1}, s_{k_2}, \ldots, s_{k_p} \), then (4) would become \( \ell(\sigma) \leq p \), which would contradict \( p < \ell(\sigma) \). So Claim 2 is proven. \( \square \)

(For details: [detnotes, Exercise 4.2(g)].)

**Cor. 5.7.** Let \( n \in \mathbb{N} \).

1. We have \( \ell(\sigma \tau) \equiv \ell(\sigma) + \ell(\tau) \mod 2 \) for all \( \sigma \in S_n \) and \( \tau \in S_n \).
2. We have \( \ell(\sigma \tau) \leq \ell(\sigma) + \ell(\tau) \) for all \( \sigma \in S_n \) and \( \tau \in S_n \).
3. If \( \sigma = s_{k_1} \circ s_{k_2} \circ \ldots \circ s_{k_q} \), then \( q \equiv \ell(\sigma) \mod 2 \).

**Proof.** [detnotes, Exercises 4.2 and 4.3] \( \square \)

**Prop. 5.8.** Let \( n \in \mathbb{N} \).

1. We have \( \ell(s_k) = 1 \) for any \( k \in [n-1] \).
2. We have \( \ell(t_{i,j}) = 2 |i-j| - 1 \) for any \( i \neq j \) in \( [n] \).
3. We have \( \ell(\text{cyc}_{i_1, i_2, \ldots, i_k}) = k - 1 \) for any \( i_1, \ldots, i_k \) distinct.
4. We have \( \ell(\text{cyc}_{i_1, i_2, \ldots, i_k}) \geq k - 1 \) for any \( i_1, i_2, \ldots, i_k \) distinct.
Proof. (a) follows from (b),
(b) is [dehnotes, Exercise 4.10],
(c),(d) are [dehnotes, Exercise 4.16]. □

Prop. 5.9. Let \( n \in \mathbb{N} \). Then,
\[
\sum_{\omega \in S_n} x^\ell(\omega) = (1 + x)(1 + x + x^2)(1 + x + x^2 + x^3) \ldots \\
(1 + x + x^2 + \ldots + x^{n-1})
\]
\[
= \prod_{i=1}^{n-1} (1 + x + \ldots + x^i)
\]

Proof. [dehnotes, 24.8], □

5.3. Signs

Def. The sign of a permutation \( \sigma \in S_n \) (where \( n \in \mathbb{N} \)) is \((-1)^{\ell(\sigma)}\).

It is called \((-1)^\sigma\) or \(\text{sign}(\sigma)\) or \(\text{sgn}(\sigma)\) or \(\varepsilon(\sigma)\)....
Thm. 5.10. Let \( n \in \mathbb{N} \). Then:

(a) \((-1)^{i_d} = 1\),

(b) \((-1)^{t_{i_d}} = -1 \quad \forall \ i \neq d\).

(c) \((-1)^{\text{cyc}_{i_1, i_2, \ldots, i_k}} = (-1)^{k-1} \quad \forall \ i_1, i_2, \ldots, i_k \text{ distinct}\).

(d) \((-1)^{o \circ \tau} = (-1)^{o} (-1)^{\tau} \quad \forall \ o, \tau \in S_n\).

(e) \((-1)^{o^{-1}} = (-1)^{o} \quad \forall \ o \in S_n\).

(f) \((-1)^{o \circ o^{-1}} = (-1)^{\tau} \quad \forall \ o, \tau \in S_n\).

(g) \((-1)^{o} = \prod_{1 \leq i < j \leq n} \frac{o(i) - o(j)}{i - j} \quad \forall \ o \in S_n\).

(h) If \( x_1, x_2, \ldots, x_n \) are any \( n \) numbers, and \( o \in S_n \), then

\[
\prod_{1 \leq i < j \leq n} (x_{o(i)} - x_{o(j)}) = (-1)^{o} \prod_{1 \leq i < j \leq n} (x_i - x_j).
\]

**Proof.**

(a) \((-1)^{i_d} = (-1)^{l(id)} = (-1)^{o} = 1\).

(b) Prop. 5.8 (b) yields: \( l(t_{i_d}) \) is odd.
(d) \((-1)^{\sigma \tau} = (-1)^{l(\sigma \tau)} = (-1)^{l(\sigma) + l(\tau)}\)

(since Cor. 5.7 (2) yields \(l(\sigma \tau) \equiv l(\sigma) + l(\tau) \mod 2\))

\[ = (-1)^{l(\sigma)} (-1)^{l(\tau)} = (-1)^{\sigma} (-1)^{\tau}, \]

(e) \((-1)^{\sigma^{-1}} = (-1)^{l(\sigma^{-1})} = \# (-1)^{l(\sigma)} \quad \text{by Prop. 5.4}\)

\[ = (-1)^{\sigma}. \]

(f) \((-1)^{\sigma \tau \sigma^{-1}} = (-1)^{l(\sigma \tau \sigma^{-1})} = (-1)^{l(\sigma)} (-1)^{l(\tau)} (-1)^{l(\sigma^{-1})}\)

\[ = (-1)^{\sigma} (-1)^{\tau} (-1)^{\sigma^{-1}} \]

\[ \overset{(d)}{=} (-1)^{\sigma \tau} (-1)^{\sigma^{-1}} \]

\[ = (-1)^{\sigma} (-1)^{\sigma^{-1}} \]

\[ \overset{(e)}{=} (-1)^{\sigma} \]

\[ = (-1)^{\sigma} (-1)^{\tau} = (-1)^{\sigma \tau}. \]

(g), (h) see [detnres, Exercise 4.13].

(c) Recall from last time:

\[ \text{cyc}_{i_1, i_2, \ldots, i_k} = t_{i_1} i_2 t_{i_2} i_3 \cdots t_{i_{k-1}} i_k. \]
Hence, \( \text{cyc}_{i_1, i_2, \ldots, i_k} = (-1)^{t_{i_1} i_2 \cdot t_{i_2} i_3 \cdots t_{i_{k-1}} i_k} \)

\[
(-1) t_{i_1} i_2 (-1) t_{i_2} i_3 \cdots (-1) t_{i_{k-1}} i_k
\]

(since (d) & induction yield

\[
(-1)^{o_1 o_2 \cdots o_p} = (-1)^{o_2} (-1)^{o_2} \cdots (-1)^{o_p}
\]

\( \forall o_1 o_2 \cdots o_p \in S_n \).

\[
(b) \quad (-1) (-1) \cdots (-1) = (-1)^{k-1}. \quad \square
\]

**Def.** Let \( n \in \mathbb{N} \). A permutation \( \sigma \in S_n \) is **called even** if \((-1)^{\sigma} = 1\) (i.e., if \( l(\sigma) \) is even), and **odd** if \((-1)^{\sigma} = -1\) (i.e., if \( l(\sigma) \) is odd).

**Cor. 5.11.** Let \( n \geq 2 \). Then,

\[
\text{(\# of even } \sigma \in S_n) = \frac{n!}{2}.
\]

**Proof.** The map
\{\text{even } \sigma \in S_n\} \rightarrow \{\text{odd } \sigma \in S_n\},
\sigma \rightarrow \sigma \circ s_2

is a bijection.

5.4. Cycle Decomposition

Example: Let \( \sigma = [4, 6, 1, 3, 5, 2, 9, 8, 7] \in S_9 \). Its cycle digraph is

\[\begin{array}{c|c|c|c|c}
& 4 & 6 & 7 & 8 \\
1 & \searrow & \nearrow & \nearrow & \nearrow \\
3 & \nearrow & \nearrow & \searrow & \nearrow \\
& \text{cyc}_{1,4,3} & \text{cyc}_{2,6} & \text{cyc}_{5} (=\text{id}) & \text{cyc}_{7,9} (=\text{id}) \\
\end{array}\]

\[\Rightarrow \quad \sigma = \text{cyc}_{1,4,3} \circ \text{cyc}_{2,6} \circ \text{cyc}_{5} \circ \text{cyc}_{7,9} \circ \text{cyc}_{8}
\]

(since the LHS and the RHS act on each \(\text{re } [9] \) in the same way).
Likewise, let \( \tau = [3, 4, 6, 2, 7, 8, 5, 1] \in S_8 \). Its cycle digraph is:

\[
\begin{array}{c|c|c}
3 & 4 & 5 \\
6 & 2 & 7 \\
8 & & \\
\text{cyc}_{1, 3, 6, 8} & \text{cyc}_{2, 4} & \text{cyc}_{5, 7}
\end{array}
\]

\[\Rightarrow \quad \tau = \text{cyc}_{1, 3, 6, 8} \circ \text{cyc}_{2, 4} \circ \text{cyc}_{5, 7} \]

Similarly, each \( \pi \in S_n \) can be written as a composition of cycles \( \text{cyc}_{i_1, i_2, \ldots, i_k} \) with each element of \( \mathbb{Z}_n \) appearing in exactly 1 of these cycles. This representation of \( \pi \) is unique up to swapping the cycles and "cycling each cycle" (\( \text{cyc}_{i_1, i_2, \ldots, i_k} = \text{cyc}_{i_2, \ldots, i_k, i_1} = \ldots = \text{cyc}_{i_k, i_1, \ldots, i_{k-1}} \)).
Thm. 5.12. Let \( \sigma \) be a permutation of a finite set \( X \),

(2) There is a list \((a_{1,1}, a_{2,1}, \ldots, a_{1,n_1}), (a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}), \ldots, (a_{k,1}, a_{k,2}, \ldots, a_{k,n_k})\) of lists of elements of \( X \), such that:
- each element of \( X \) appears exactly once in \( a_{i,1}, a_{i,2}, \ldots, a_{i,n_i} \)
- \( \sigma = \text{cyc } a_{1,1}, a_{1,2}, \ldots, a_{1,n_1} \text{ cyc } a_{2,1}, a_{2,2}, \ldots, a_{2,n_2} \text{ cyc } a_{3,1}, a_{3,2}, \ldots, a_{3,n_3} \ldots \text{ cyc } a_{k,1}, a_{k,2}, \ldots, a_{k,n_k} \)

Such a list is called a disjoint cycle decomposition of \( \sigma \).
(b) Any two such lists can be obtained from each other by swapping sublists & cycling each sublist.

(c) If we additionally require

- $a_{x,1} > a_{2,1} > \ldots > a_{R,1}$

- $a_{i,1} < a_{i,p}$ \quad $\forall i \forall p$,

then this list is unique.

Proof: [Goodman, "Algebra: Abstract & Concrete", proof of Theorem 1.5.3]. Also, see Example above.