Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

See [Fall2017-HW1s, solution to Exercise 8] for an example of how a counting proof can be written.

0.1. More on the Sierpinski triangle in Pascal’s triangle

Exercise 1. Let \( n \in \mathbb{N} \).

(a) Prove that the integer \( \binom{2^n - 1}{b} \) is odd for each \( b \in \{0, 1, \ldots, 2^n - 1\} \).

(b) Prove that the integer \( \binom{2^n}{b} \) is even for each \( b \in \{1, 2, \ldots, 2^n - 1\} \).

[Here, the set \( \{0, 1, \ldots, 2^n - 1\} \) means the set of all integers \( k \) with \( 0 \leq k \leq 2^n - 1 \), and the set \( \{1, 2, \ldots, 2^n - 1\} \) means the set of all integers \( k \) with \( 1 \leq k \leq 2^n - 1 \).]

0.2. Counting by symmetry

Recall that if \( n \in \mathbb{N} \), then \( [n] \) denotes the \( n \)-element set \( \{1, 2, \ldots, n\} \). If \( n \in \mathbb{N} \), then \( S_n \) shall mean the set of all permutations of the set \( [n] \). The number of these permutations is \( |S_n| = n! \). (We shall prove this in class soon.) Note that \( S_n \) is called the \( n \)-th symmetric group.

Proposition 0.1. Let \( n \geq 4 \) be an integer. Then, the number of all permutations \( \sigma \in S_n \) satisfying \( \sigma(3) > \sigma(4) \) is \( n! / 2 \).

Proof of Proposition 0.1

I say that a permutation \( \sigma \in S_n \) is:

- **green** if it satisfies \( \sigma(3) > \sigma(4) \);
- **red** if it satisfies \( \sigma(3) < \sigma(4) \).

Every permutation \( \sigma \in S_n \) is either green or red (indeed, every permutation \( \sigma \in S_n \) is injective, and thus satisfies \( \sigma(3) \neq \sigma(4) \), so that it must satisfy either \( \sigma(3) > \sigma(4) \) or \( \sigma(3) < \sigma(4) \)), but no permutation \( \sigma \in S_n \) can be both green and red at the same time (since \( \sigma(3) > \sigma(4) \) would contradict \( \sigma(3) < \sigma(4) \)). Hence, the set \( S_n \) is the union of its two disjoint subsets \{green permutations \( \sigma \in S_n \)\} and \{red permutations \( \sigma \in S_n \)\}. Thus,

\[
|S_n| = |\{\text{green permutations } \sigma \in S_n\}| + |\{\text{red permutations } \sigma \in S_n\}|. \tag{1}
\]
On the other hand, I claim that “the colors are equidistributed”, i.e., the number of green permutations $\sigma \in S_n$ equals the number of red permutations $\sigma \in S_n$.

To prove this, I will construct a bijection from \{green permutations $\sigma \in S_n$\} to \{red permutations $\sigma \in S_n$\}.

Indeed, let $s_3$ be the permutation of $[n]$ that swaps the numbers 3 and 4 while leaving all other numbers unchanged. That is, $s_3$ is given by

$$s_3(i) = \begin{cases} 4, & \text{if } i = 3; \\ 3, & \text{if } i = 4; \\ i, & \text{if } i \notin \{3, 4\} \end{cases} \text{ for all } i \in [n].$$

(In one-line notation, $s_3$ is represented as $(1, 2, 4, 3, 5, 6, \ldots, n)$, where only the two numbers 3 and 4 are out of order.)

Notice that $s_3 \circ s_3 = \text{id}$. (Visually speaking, this is clear: If we swap 3 and 4, and then swap 3 and 4 again, then all numbers return to their old places.)

If $\alpha$ and $\beta$ are two permutations of $[n]$, then their composition $\alpha \circ \beta$ is a permutation of $[n]$ as well. Hence, for every permutation $\sigma \in S_n$, the map $\sigma \circ s_3$ is also a permutation of $[n]$.

We now claim that

if $\sigma \in S_n$ is green, then $\sigma \circ s_3 \in S_n$ is red. \hfill (2)

[Proof of (2): Assume that $\sigma \in S_n$ is green. Thus, $\sigma(3) > \sigma(4)$ (by the definition of “green”).

We know $\sigma \circ s_3$ is a permutation of $[n]$. In other words, $\sigma \circ s_3 \in S_n$. We must prove that $\sigma \circ s_3$ is red. In other words, we must prove that $(\sigma \circ s_3)(3) < (\sigma \circ s_3)(4)$ (because this is what it means for $\sigma \circ s_3$ to be red).

But the definition of $s_3$ shows that $s_3(3) = 4$ and $s_3(4) = 3$. Thus, $(\sigma \circ s_3)(3) = \sigma(s_3(3)) = \sigma(4)$ and $(\sigma \circ s_3)(4) = \sigma(s_3(4)) = \sigma(3)$. Hence, $(\sigma \circ s_3)(4) = \sigma(3) > \sigma(4) = (\sigma \circ s_3)(3)$. In other words, $(\sigma \circ s_3)(3) < (\sigma \circ s_3)(4)$. But this is exactly what we wanted to prove. Thus, (2) is proven.]

An analogous argument shows that

if $\sigma \in S_n$ is red, then $\sigma \circ s_3 \in S_n$ is green. \hfill (3)

Now, let $\alpha$ be the map

$$\{\text{green permutations } \sigma \in S_n\} \to \{\text{red permutations } \sigma \in S_n\}, \quad \sigma \mapsto \sigma \circ s_3$$

\(^1\text{because permutations of } [n] \text{ are just bijective maps } [n] \to [n], \text{ but the composition of two bijective maps is again bijective}\)
(this is well-defined because of (2)). Let $\beta$ be the map
\[
\{ \text{red permutations } \sigma \in S_n \} \to \{ \text{green permutations } \sigma \in S_n \},
\sigma \mapsto \sigma \circ s_3
\]
(this is well-defined because of (3)). We have $\alpha \circ \beta = \text{id}$ (since every red permutation $\sigma \in S_n$ satisfies
\[
(\alpha \circ \beta) (\sigma) = \alpha \left( \beta (\sigma) \right) = \alpha (\sigma \circ s_3)
\]
(by the definition of $\beta$)
\[
= (\sigma \circ s_3) \circ s_3 = \sigma \circ (s_3 \circ s_3) = \sigma = \text{id} (\sigma)
\]
$\circ \text{id}$)
and $\beta \circ \alpha = \text{id}$ (by an analogous computation). Thus, the two maps $\alpha$ and $\beta$ are mutually inverse. Hence, $\alpha$ is a bijection. Thus, we have found a bijection from \{green permutations $\sigma \in S_n$\} to \{red permutations $\sigma \in S_n$\} (namely, $\alpha$). Therefore,
\[
|\{ \text{green permutations } \sigma \in S_n \}| = |\{ \text{red permutations } \sigma \in S_n \}|. \quad (4)
\]
Now, (1) becomes
\[
|S_n| = |\{ \text{green permutations } \sigma \in S_n \}| + |\{ \text{red permutations } \sigma \in S_n \}|
\]
\[
= |\{ \text{green permutations } \sigma \in S_n \}| + |\{ \text{green permutations } \sigma \in S_n \}|
\]
\[
= 2 \cdot |\{ \text{green permutations } \sigma \in S_n \}|.
\]
Hence,
\[
|\{ \text{green permutations } \sigma \in S_n \}| = \frac{1}{2} \cdot |S_n| = \frac{1}{2} n! = n!/2.
\]
In other words, the number of all green permutations $\sigma \in S_n$ is $n!/2$. In other words, the number of all permutations $\sigma \in S_n$ satisfying $\sigma(3) > \sigma(4)$ is $n!/2$ (because these permutations are precisely the green permutations $\sigma \in S_n$). This proves Proposition 0.1.

Our above proof was an example of a “counting by symmetry”: We did not count the green permutations directly; instead, we showed that they are in bijection with the remaining (i.e., red) permutations $\sigma \in S_n$ (that is, we matched up each green permutation with a red one), from which we concluded that they make up exactly half of the set $S_n$; and this told us that there are $\frac{1}{2} |S_n| = n!/2$ of them.
Exercise 2. Let $n \geq 4$ be an integer. Prove the following:

(a) The number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2)$ and $\sigma(3) > \sigma(4)$ is $n!/4$.

(b) The number of all permutations $\sigma \in S_n$ satisfying $\sigma(1) > \sigma(2) > \sigma(3)$ is $n!/6$.

[Hint: You’ll need more than 2 colors...]

0.3. More on Fibonacci numbers

Recall that the Fibonacci sequence is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2.$$  

Exercise 3. Prove the following:

(a) We have $7f_n = f_{n-4} + f_{n+4}$ for each $n \geq 4$.

(b) We have $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ for each $n \in \mathbb{N}$.

(c) We have $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ for each $n \in \mathbb{N}$.

(d) We have $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$ for each $n \in \mathbb{N}$.

(e) We have $f_{m+n+1} = f_{m+1}f_{n+1} + f_mf_n$ for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(f) For every $m \in \mathbb{N}$, we have

$$f_{2m+2} = \sum_{(a,b) \in \mathbb{N}^2; \ a+b \leq m} \binom{m-a}{b} \binom{m-b}{a}.$$  

[Hint: All parts can be proven bijectively; part (f) is actually easiest to prove bijectively! (On the other hand, proving part (a) bijectively is a challenge; there are much easier ways.) As a reminder: Any exercises from previous problem sets can be used without proof.]

0.4. More lattice path counting

If $(a, b) \in \mathbb{Z}^2$ and $(c, d) \in \mathbb{Z}^2$ are two points on the integer lattice, then a lattice path from $(a, b)$ to $(c, d)$ is a path from $(a, b)$ to $(c, d)$ that uses only two kinds of steps:

- up-steps (U), which have the form $(x, y) \mapsto (x, y+1)$;
- right-steps (R), which have the form $(x, y) \mapsto (x+1, y)$.

Thus, strictly speaking, a lattice path from $(a, b)$ to $(c, d)$ is a sequence $(v_0, v_1, \ldots, v_n)$ of points $v_i \in \mathbb{Z}^2$ such that for each $i \in [n]$, the difference vector $v_i - v_{i-1}$ is either $(0,1)$ or $(1,0)$.
Exercise 4. (a) Given six integers $a_1, b_1, c_1, a_2, b_2, c_2$ satisfying $0 \leq a_1 \leq b_1 \leq c_1$ and $0 \leq a_2 \leq b_2 \leq c_2$. How many lattice paths from $(0,0)$ to $(c_1, c_2)$ pass through none of the points $(a_1, a_2)$ nor $(b_1, b_2)$?

(b) Given six integers $a, b, c, A, B, C$ satisfying $0 \leq a \leq b \leq c$ and $0 \leq A \leq B \leq C$. How many $c$-element subsets $S$ of $[C]$ satisfy $|S \cap [A]| \neq a$ and $|S \cap [B]| \neq b$?

0.5. Zig-zag binary strings

If $n \in \mathbb{N}$, then a binary $n$-string shall mean an $n$-tuple of elements of $\{0,1\}$. (For example, $(0,1,0,1)$ is a binary 4-string.)

We say that a binary $n$-string $(a_1, a_2, \ldots, a_n)$ is zig-zag if it satisfies $a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots$ (in other words, $a_i \leq a_{i+1}$ for every odd $i \in [n-1]$, and $a_i \geq a_{i+1}$ for every even $i \in [n-1]$).

For example, $(0,1,1,0,1,0,0,1)$ is a zig-zag binary 8-string, but $(0,1,0,0,1)$ is not.

Exercise 5. Find a simple expression (no summation signs, only known functions and sequences) for the number of zig-zag binary $n$-strings for all $n \in \mathbb{N}$.

0.6. A binomial identity

Exercise 6. Let $n \in \mathbb{N}$. Prove that

$$
\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} = 2 \cdot \frac{n+1}{n+2} [n \text{ is even}].
$$

(Again, we are using the Iverson bracket notation, so $[n \text{ is even}]$ is 1 if $n$ is even and 0 otherwise.)

[Hint: Show that $\frac{1}{\binom{n}{k}} = \left( \frac{1}{\binom{n+1}{k}} + \frac{1}{\binom{n+1}{k+1}} \right) \frac{n+1}{n+2}$ for each $k \in \{0,1,\ldots,n\}$.]

Remark 0.2. The left hand side in Exercise 6 is the alternating sum of the reciprocals of all (nonzero) binomial coefficients in the $n$-th row of Pascal’s triangle. What about the regular (non-alternating) sum? It appears that the simplest known formula merely rewrites it as a different (somewhat simpler) sum:

$$
\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} 2^k.
$$
See, e.g., https://math.stackexchange.com/a/481686/ for a proof of this formula (and also of the fact that the sum on the left tends to 2 as \( n \to \infty \)).

0.7. Splitting integers into binomial coefficients

Exercise 7. Let \( j \) be a positive integer. A \( j \)-trail shall mean a \( j \)-tuple \((n_1, n_2, \ldots, n_j)\) of nonnegative integers satisfying \( n_1 < n_2 < \cdots < n_j \).

Let \( n \in \mathbb{N} \). Prove that there exists a unique \( j \)-trail \((n_1, n_2, \ldots, n_j)\) satisfying

\[
n = \sum_{k=1}^{j} \binom{n_k}{k}.
\]

Example 0.3. For \( j = 3 \), Exercise 7 says the following: For each \( n \in \mathbb{N} \), there exists a unique 3-trail \((n_1, n_2, n_3)\) satisfying

\[
n = \binom{n_1}{1} + \binom{n_2}{2} + \binom{n_3}{3}.
\]

For example, for \( n = 0 \), this 3-trail is \((0, 1, 2)\); for \( n = 1 \), this 3-trail is \((0, 1, 3)\); for \( n = 5 \), this 3-trail is \((0, 2, 4)\) (since \( 5 = \binom{0}{1} + \binom{2}{2} + \binom{4}{3} \)).

References
