1 Exercise 1

1.1 Problem

Let \( n \in \mathbb{N} \).

(a) Prove that the integer \( \binom{2n - 1}{b} \) is odd for each \( b \in \{0, 1, \ldots, 2^n - 1\} \).

(b) Prove that the integer \( \binom{2n}{b} \) is even for each \( b \in \{1, 2, \ldots, 2^n - 1\} \).

[Here, the set \( \{0, 1, \ldots, 2^n - 1\} \) means the set of all integers \( k \) with \( 0 \leq k \leq 2^n - 1 \), and the set \( \{1, 2, \ldots, 2^n - 1\} \) means the set of all integers \( k \) with \( 1 \leq k \leq 2^n - 1 \).]

1.2 Solution

Lemma 1.1. The definition of binomial coefficient demonstrates that for any \( k \in \mathbb{N} \)

\[
\binom{0}{k} = [k = 0].
\]

This was shown briefly in class.

We now solve the actual exercise.
(a) Proof. Let \( p(n) \) be the logical statement that (a) holds for some \( n \in \mathbb{N} \). First, for \( n = 0 \) we have \( b \in \{0, \ldots, 2^0 - 1\} = \{0\} \) so
\[
\binom{2^n - 1}{b} = \binom{0}{0} = 1,
\]
which is clearly odd, so \( p(0) \) is true.

For an inductive hypothesis, assume \( p(m) \) holds for some \( m \in \mathbb{N} \). Let’s examine \( p(m + 1) \). We can express \( 2^{m+1} - 1 \) as \( 2^m + (2^m - 1) \). Clearly \( 2^m - 1 \) is in \( \mathbb{N} \) since \( 2^m \in \mathbb{N} \) and \( 2^m \geq 1 \). Fix \( b \in \{0, 1, \ldots, 2^m + 1 - 1\} \).

If \( b \leq 2^m - 1 \), we are able to apply \( \text{HW1} \) Exe4 (congruence 1) substituting \( n := m, a := 2^m - 1, b := b \) since our variables satisfy the domain constraints (since \( b \in \{0, 1, \ldots, 2^m - 1\} \)), and obtain
\[
\binom{2^{m+1} - 1}{b} = \binom{2^m + 2^m - 1}{b} \equiv \binom{2^m - 1}{b} \mod 2.
\]

If \( b > 2^m - 1 \), we are able to apply \( \text{HW1} \) Exe4 (congruence 2) substituting \( n := m, a := 2^m - 1, b := b - 2^m \) where our variables again satisfy the domain constraints (since \( b > 2^m - 1 \) yields \( b - 2^m \geq 0 \) and thus \( b - 2^m \in \{0, 1, \ldots, 2^m - 1\} \)), and obtain
\[
\binom{2^{m+1} - 1}{b} = \binom{2^m + 2^m - 1}{2^m + b - 2^m} \equiv \binom{2^m - 1}{b} \mod 2.
\]

In either case, we obtain
\[
\binom{2^{m+1} - 1}{b} \equiv \binom{2^m - 1}{b} \mod 2.
\]

From our inductive hypothesis, we know that \( \binom{2^m - 1}{b} \) is odd, so \( \binom{2^{m+1} - 1}{b} \) is also odd since they are congruent modulo 2. Thus, \( p(m + 1) \) holds given \( p(m) \). Hence, \( p(n) \) holds for all \( n \in \mathbb{N} \) via the Principle of Mathematical Induction. \( \square \)

(b) Proof. Both 0 and \( b \) belong to \( \{0, 1, \ldots, 2^m - 1\} \). Hence, we can apply \( \text{HW1} \) Exe4 (congruence 2) directly (substituting \( n := n, a := 0, b := b \)) to get
\[
\binom{2^m}{b} \equiv \binom{0}{b} \mod 2 \tag{Lemma 1.1}
\]

So \( \binom{2^m}{b} \) is even. \( \square \)

Remark 1.2. For (b), note that the set \( \{1, 2, \ldots, 2^m - 1\} \) is the empty set for \( n = 0 \). This means that (b) is vacuously true for \( n = 0 \), since no there are no \( b \) that it would make a statement about.

2 Exercise 4

2.1 Problem

(a) Given six integers \( a_1, b_1, c_1, a_2, b_2, c_2 \) satisfying \( 0 \leq a_1 \leq b_1 \leq c_1 \) and \( 0 \leq a_2 \leq b_2 \leq c_2 \). How many lattice paths from \((0, 0)\) to \((c_1, c_2)\) pass through none of the points \((a_1, a_2)\) nor \((b_1, b_2)\)?

(b) Given six integers \( a, b, c, A, B, C \) satisfying \( 0 \leq a \leq b \leq c \) and \( 0 \leq A \leq B \leq C \). How many \( c \)-element subsets \( S \) of \( [C] \) satisfy \( |S \cap [A]| \neq a \) and \( |S \cap [B]| \neq b \)?
Lemma 2.1. Let $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ be such that $m + n \geq 0$. The number of lattice paths from $(0,0)$ to $(m,n)$ is $\binom{m+n}{n} = \binom{m+n}{m}$.

Proof. In the case when $m \in \mathbb{N}$ and $n \in \mathbb{N}$, this was proven in class. Thus, it remains to consider the other case. So, we assume that one of $m$ and $n$ does not belong to $\mathbb{N}$. In other words, one of $m$ and $n$ is negative. Then, the other one is smaller than $m+n$. Hence, both $\binom{m+n}{n}$ and $\binom{m+n}{m}$ equal 0 (since $m+n \in \mathbb{N}$). It remains to check that the number of lattice paths from $(0,0)$ to $(m,n)$ is 0 as well. But this is clear: Since one of $m$ and $n$ is negative, there are no lattice paths from $(0,0)$ to $(m,n)$ (because both coordinates can only increase along a lattice path, and therefore we can never get to negative coordinates if we start at $(0,0)$).

We now solve the exercise:

(a) Rather than just solving the exercise as it is stated, we generalize it a little bit: We replace the requirement “$0 \leq a_1 \leq b_1 \leq c_1$ and “$0 \leq a_2 \leq b_2 \leq c_2$” by the (weaker) requirement “$0 \leq a_1 + a_2 \leq b_1 + b_2 \leq c_1 + c_2$”. This will come in handy when we later deduce part (b) from part (a).

Proof. Define the following sets

$$U = \{ \text{all lattice paths from } (0,0) \text{ to } (c_1,c_2) \},$$

$$P_a = \{ \text{paths from } (0,0) \text{ to } (c_1,c_2) \text{ passing through } (a_1,a_2) \},$$

$$P_b = \{ \text{paths from } (0,0) \text{ to } (c_1,c_2) \text{ passing through } (b_1,b_2) \};$$

thus,

$$U \setminus (P_a \cup P_b) = \{ \text{paths from } (0,0) \text{ to } (c_1,c_2) \text{ passing through neither } (a_1,a_2) \text{ nor } (b_1,b_2) \}.$$

But the Inclusion-Exclusion Principle shows that

$$|U \setminus (P_a \cup P_b)| = |U| - |P_a| - |P_b| + |P_a \cap P_b|.$$

So, we just need to count each of these sets using Lemma 2.1. If $m, n, p, q$ are integers satisfying $0 \leq p + q \leq m + n$, then the number of paths from $(0,0)$ to $(m,n)$ passing through $(p,q)$ is $\binom{m+n}{p} \binom{(m+n)-(p+q)}{m-p}$ (since we are independently choosing paths from $(0,0)$ to $(p,q)$ and from $(p,q)$ to $(m,n)$, but Lemma 2.1 shows that there are $\binom{p+q}{p}$ options for the former and $\binom{(m+n)-(p+q)}{m-p}$ options for the latter). This generalizes to a formula for the number of paths from $(0,0)$ to $(m,n)$ passing through $k$ specified points; it is expressed as a product of $k+1$ binomial coefficients.
(indeed, any lattice path must traverse the \(k\) points in the order of increasing sum of coordinates\(^1\) so it breaks into \(k + 1\) smaller paths with known endpoints). Hence,

\[
|P_a| = \left(\frac{a_1 + a_2}{a_1}\right) \frac{\left(\frac{c_1 + c_2}{c_1} \right) - \left(\frac{a_1 + a_2}{a_1}\right)}{c_1 - a_1};
\]

\[
|P_b| = \left(\frac{b_1 + b_2}{b_1}\right) \frac{\left(\frac{c_1 + c_2}{c_1} \right) - \left(\frac{b_1 + b_2}{b_1}\right)}{c_1 - b_1};
\]

\[
|P_a \cap P_b| = \left(\frac{a_1 + a_2}{a_1}\right) \frac{\left(\frac{b_1 + b_2}{b_1} \right) - \left(\frac{a_1 + a_2}{a_1}\right)}{b_1 - a_1} \frac{\left(\frac{c_1 + c_2}{c_1} \right) - \left(\frac{b_1 + b_2}{b_1}\right)}{c_1 - b_1}.
\]

We can then count the desired lattice paths:

\[
|U \setminus (P_a \cup P_b)| = |U| - |P_a| - |P_b| + |P_a \cap P_b|
\]

\[
= \left(\frac{c_1 + c_2}{c_1}\right) - \left(\frac{a_1 + a_2}{a_1}\right) \frac{\left(\frac{c_1 + c_2}{c_1} \right) - \left(\frac{a_1 + a_2}{a_1}\right)}{c_1 - a_1}
\]

\[
- \left(\frac{b_1 + b_2}{b_1}\right) \frac{\left(\frac{c_1 + c_2}{c_1} \right) - \left(\frac{b_1 + b_2}{b_1}\right)}{c_1 - b_1}
\]

\[
+ \left(\frac{a_1 + a_2}{a_1}\right) \frac{\left(\frac{b_1 + b_2}{b_1} \right) - \left(\frac{a_1 + a_2}{a_1}\right)}{b_1 - a_1} \frac{\left(\frac{c_1 + c_2}{c_1} \right) - \left(\frac{b_1 + b_2}{b_1}\right)}{c_1 - b_1}.
\]

\(\Box\)

(b) **Proof.** Let \(a_1 = a, b_1 = b,\) and \(c_1 = c.\)

Let \(a_2 = A - a, b_2 = B - b,\) and \(c_2 = C - c.\)

Clearly, \(0 \leq a_1 + a_2 \leq b_1 + b_2 \leq c_1 + c_2.\)

Let \(P_c\) be the set \(U \setminus (P_a \cup P_b)\) from (a).

Let us construct a bijection between \(P_c\) and \(Q_c,\) the set of all \(c\)-element subsets \(S\) of \([C]\) satisfy \(|S \cap [A]| \neq a\) and \(|S \cap [B]| \neq b.\)

Every path in \(P_c\) can be bijectively mapped to a subset \(S\) of \([C]\) with \(|S| = c.\) For a given path \(P \in P_c,\) it has \(C = c_1 + c_2\) steps, either north or east. Let us number these steps \(\{1, 2, \ldots, C\} = [C].\) Define a subset \(S \subset [C]\) as the set of the indices of eastern steps (so a path from \((0, 0)\) to \((1, 1)\) consisting of an eastern step followed by a northern step would have \(S = \{1\}\)). This constructed set \(S\) is clearly a subset of \([C]\) containing \(c = c_1\) elements. Since \(P\) does not pass through \((a_1, a_2),\) \(P\) either takes more or less than \(a_1\) eastern steps in its first \(A = a_1 + a_2\) steps. This implies that there are either more or less than \(a_1\) elements in the first \(A\) elements of \(S,\) so \(|S \cap [A]| \neq a.\) Similarly, since \(P\) does not pass through \((b_1, b_2),\) we have \(|S \cap [B]| \neq b.\)

Thus, \(S\) is a valid member of \(Q_c.\)

The inverse of this function \((Q_c \rightarrow P_c)\) follows naturally: For a given subset \(S \in Q_c,\) map each element \(i \in S\) to an eastern step in a path \(P\) and each element \(j \in [C] \setminus S\) to a northern step. A matching argument can be given to the correctness of this mapping. Moreover, this is a bijection since it is a two-sided inverse and each path \(P\) uniquely determines a subset \(S\) and vice versa.

Thus, there exists a bijection between \(P_c\) and \(Q_c,\) so that \(|Q_c| = |P_c|\). The right hand side has been expressed in (a).

\(\Box\)

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\(^1\)If two of the \(k\) points have the same sum of coordinates, while being distinct, then the number is 0, because no lattice path can traverse them both.
3 Exercise 5

3.1 Problem
We say that a binary $n$-string $(a_1, a_2, \ldots, a_n)$ is zig-zag if it satisfies $a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots$ (in other words, $a_i \leq a_{i+1}$ for every odd $i \in [n-1]$, and $a_i \geq a_{i+1}$ for every even $i \in [n-1]$).
Find a simple expression (no summation signs, only known functions and sequences) for the number of zig-zag binary $n$-strings for all $n \in \mathbb{N}$.

3.2 Solution
We shall use the ceiling function: For any integer $x$, we let $\lceil x \rceil$ denote the smallest integer that is $\geq x$.

First let us define the color of a binary string $b_n$ as

$$\text{color}(b_n) = \begin{cases} 
\text{black} & \text{if } b_n \text{'s final element is 0} \\
\text{red} & \text{otherwise}
\end{cases}$$

(in particular, we count the binary 0-string () as red). Let $z_{n,\text{black}}$ and $z_{n,\text{red}}$ denote the number of black and red zig-zag binary $n$-strings, respectively. Let $z_n$ be the number of all zig-zag binary $n$-strings. Then, $z_n = z_{n,\text{black}} + z_{n,\text{red}}$, since every zig-zag binary string must either be black or red.

If $b$ is a binary $n$-string, and if $g \in \{0, 1\}$, then $b \# g$ will mean the binary $(n + 1)$-string obtained by appending $g$ to the end of $b$. (For example, $(0, 1, 1, 0) \# 1 = (0, 1, 1, 0, 1)$.)

Let us devise a constructive algorithm to recursively build zig-zag binary strings. Given a zig-zag binary $n$-string $b_n$, the following algorithm constructs all zig-zag binary $(n + 1)$-strings that begin with $b_n$:

\begin{algorithm}
\caption{Generating zig-zag binary strings}
\begin{algorithmic}[1]
\Procedure{GenerateZigZag}{b_n}
\If{$b_n$ is black}
\If{$n$ is even} \Return{$b_n \# 0$}
\Else \Return{$b_n \# 0, b_n \# 1$}
\EndIf
\Else \% $b_n$ is red \EndIf
\If{$n$ is even} \Return{$b_n \# 0, b_n \# 1$}
\Else \EndIf
\EndProcedure
\end{algorithmic}
\end{algorithm}

This algorithm can be shown to be correct and that it generates all zig-zag binary $(n + 1)$-strings. Simply, it generates all binary $(n + 1)$-strings but prunes those that do not satisfy the zig-zag condition; lines 3 and 7 exclude strings which do not satisfy $a_i \leq a_{i+1}$ for every odd $i \in [n]$, and $a_i \geq a_{i+1}$ for every even $i \in [n]$.

Applying Algorithm 3.1 for low $n$ gives familiar values for $z_{n,\text{black}}$, $z_{n,\text{red}}$, and $z_n$.

We now solve the exercise:

\textbf{Proof.} Define $p(n)$ as the logical statement, for $n \in \mathbb{N}$, that

$$z_{n,\text{black}} = f_{2\lceil \frac{n}{2} \rceil} \quad \text{and} \quad z_{n,\text{red}} = f_{2\lceil \frac{n+1}{2} \rceil - 1},$$

where $f_i$ is the $i^{th}$ Fibonacci number (having $f_0 = 0, f_1 = 1$). For the base case, $n = 0$, we have one red zig-zag binary string and no black zig-zag binary strings, so $p(0)$ holds since $z_{n,\text{black}} = f_{2\lceil \frac{2}{2} \rceil} = f_0 = 0$ and $z_{n,\text{red}} = f_{2\lceil \frac{0+1}{2} \rceil - 1} = f_1 = 1$. 

Brady Olson (edited by Darij Grinberg), — 5 —
For the inductive hypothesis, assume $p(m)$ holds for some $m \in \mathbb{N}$. Then, let’s examine $p(m + 1)$; count the red and black zig-zag binary $(m + 1)$-strings.

If $m$ is even, for each black zig-zag binary $m$-string, $\text{GenerateZigZag}$ will generate 1 black zig-zag binary $(m + 1)$-string (line 3). Likewise, each red zig-zag binary $m$-string will generate 1 black zig-zag binary $(m + 1)$-string and 1 red zig-zag binary $(m + 1)$-string (line 6). So, there are $z_{m,\text{black}} + z_{m,\text{red}}$ black zig-zag binary $(m + 1)$-strings and $z_{m,\text{red}}$ red zig-zag binary $(m + 1)$-strings. This gives us the expressions

$$z_{m+1,\text{black}} = z_{m,\text{black}} + z_{m,\text{red}} = f\left(\frac{m}{2}\right) + f\left(\frac{m+1}{2}\right) - 1$$

(the symbol “$\equiv$” means “equals, by the inductive hypothesis”). Since $m$ is even, this evaluates to

$$z_{m+1,\text{black}} = f_m + f_{m+2} = f_{m+2} = f\left(\frac{m+2}{2}\right),$$

where the last equality is true since $m$ is even. More simply, the number of red zig-zag binary $(m + 1)$-strings is the same as the number of those with length $m$:

$$z_{m+1,\text{red}} = z_{m,\text{red}} = f\left(\frac{m+1}{2}\right) = f\left(\frac{m+1}{2}\right) - 1,$$

where again the last equality holds since $m$ is even.

On the other hand, if $m$ is odd, for each black zig-zag binary $m$-string, $\text{GenerateZigZag}$ will generate 1 black zig-zag binary $(m + 1)$-string and 1 red zig-zag binary $(m + 1)$-string (line 4). Likewise, each red zig-zag binary $m$-string will generate 1 red zig-zag binary $(m + 1)$-string (line 7). So, there are $z_{m,\text{black}} + z_{m,\text{red}}$ red zig-zag binary $(m + 1)$-strings and $z_{m,\text{black}}$ black zig-zag binary $(m + 1)$-strings. A very similar argument as with even $m$ (where ceiling terms are affected differently since $m$ is odd) yields the same results.

So, $p(m + 1)$ holds given $p(m)$. Hence, by the Principle of Mathematical Induction, $p(n)$ holds for all $n \in \mathbb{N}$.

Now, recall that $z_n = z_{n,\text{black}} + z_{n,\text{red}}$. Since $p(n)$ holds, this simplifies to

$$z_n = f\left(\frac{n}{2}\right) + f\left(\frac{n+1}{2}\right) - 1 = \begin{cases} f_n + f_{n+1} & \text{if } n \text{ is even} \\ f_{n+1} + f_n & \text{if } n \text{ is odd} \end{cases} = f_{n+2}. $$

So, the number of zig-zag binary $n$-strings is $z_n = f_{n+2}$, the $(n + 2)^{th}$ Fibonacci number.

\[\square\]