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Please write your name on each page. Feel free to use LaTeX (here is a sample file with lots of amenities included).

Recall the following:

• If \( n \in \mathbb{N} \), then \([n]\) denotes the \( n \)-element set \( \{1, 2, \ldots, n\} \).

• We use the Iverson bracket notation.

• If \( a \in \mathbb{N} \) and \( b \in \mathbb{N} \), then \( \text{sur}(a,b) \) denotes the number of surjective maps from \([a]\) to \([b]\).

0.1. Counting first-even tuples

**Exercise 1.** Let \( n \) and \( d \) be two positive integers.
An \( n \)-tuple \((x_1, x_2, \ldots, x_n) \in [d]^n\) will be called first-even if its first entry \( x_1 \) occurs in it an even number of times (i.e., the number of \( i \in [n] \) satisfying \( x_i = x_1 \) is even). (For example, the 3-tuples \((1, 5, 1)\) and \((2, 2, 3)\) are first-even, while the 3-tuple \((4, 1, 1)\) is not.)

Prove that the number of first-even \( n \)-tuples in \([d]^n\) is \( \frac{1}{2}d \left(d^{n-1} - (d-2)^{n-1}\right) \).

0.2. Counting legal paths (generalization of Catalan numbers)

Recall the notion of a lattice path, defined in Midterm 1. (Lattice paths have up-steps and right-steps.)

We say that a point \((x, y) \in \mathbb{Z}^2\) is off-limits if \( y > x \). (Thus, the off-limits points are the ones that lie strictly above the \( x = y \) diagonal in Cartesian coordinates.)

A lattice path \((v_0, v_1, \ldots, v_n)\) is said to be legal if none of the points \(v_0, v_1, \ldots, v_n\) is off-limits.
For example, the lattice path drawn from $(0,0)$ to $(4,5)$ drawn in the picture is not legal, since it contains the off-limits point $(3,4)$. Meanwhile, the lattice path from $(0,0)$ to $(4,4)$ drawn in the picture is legal.

For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we let $L_{n,m}$ be the number of all legal lattice paths from $(0,0)$ to $(n,m)$. Clearly, $L_{n,m} = 0$ if any of $n$ and $m$ is negative. Also, $L_{n,m} = 0$ if $m > n$ (because if $m > n$, then the point $(n,m)$ is off-limits).

**Exercise 2.** (a) Prove that $L_{n,m} = L_{n-1,m} + L_{n,m-1}$ for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n \geq m$ and $(n,m) \neq (0,0)$.

(b) Prove that

$$L_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m - 1$.

[The requirement $n \geq m - 1$ as opposed to $n \geq m$ is not a typo; the equality still holds for $n = m - 1$, albeit for fairly simple reasons.]

(c) Prove that $L_{n,m} = \frac{n+1-m}{n+1} \binom{n+m}{m}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $n \geq m - 1$.

(d) Prove that $L_{n,n} = \frac{1}{n+1} \binom{2n}{n}$ for any $n \in \mathbb{N}$.

1Formally speaking, this lattice path is the list

$((0,0), (1,0), (1,1), (2,1), (3,1), (3,2), (3,3), (3,4), (4,4), (5,4))$. 
0.3. Scary fractions

**Exercise 3.** Let $k$, $a$ and $b$ be three positive integers such that $k \leq a \leq b$. Prove that

$$\frac{k-1}{k} \sum_{n=a}^{b} \frac{1}{\binom{n}{k}} = \frac{1}{\binom{a-1}{k-1}} - \frac{1}{\binom{b}{k-1}}.$$

0.4. Derangements that are involutions

**Definition 0.1.** Let $\sigma$ be a permutation of a set $X$.

(a) We say that $\sigma$ is a derangement if and only if each $x \in X$ satisfies $\sigma(x) \neq x$.

(b) We say that $\sigma$ is an involution if and only if $\sigma \circ \sigma = \text{id}$ (that is, each $x \in X$ satisfies $\sigma(\sigma(x)) = x$).

For example, the permutation $\alpha$ of the set $[5]$ that sends $1, 2, 3, 4, 5$ to $3, 5, 1, 4, 2$ is an involution (it satisfies $\alpha \left(\frac{\alpha(1)}{3}\right) = \alpha \left(\frac{\alpha(2)}{5}\right) = 1$ and $\alpha(3) = 2$ and similarly $\alpha(\alpha(x)) = x$ for all other $x \in [5]$), but not a derangement (since $\alpha(4) = 4$).

On the other hand, the permutation $\beta$ of the set $[6]$ that sends $1, 2, 3, 4, 5, 6$ to $3, 4, 2, 1, 6, 5$ is a derangement (it satisfies $\beta(x) \neq x$ for all $x \in [4]$), but not an involution (since $\beta(\beta(1)) \neq 1$).

**Exercise 4.** Let $n \in \mathbb{N}$. Prove the following:

(a) If $n$ is odd, then there exist no derangements of $[n]$ that are involutions.

(b) If $n$ is even, then the number of derangements of $[n]$ that are involutions is $n! \cdot \frac{2^{n/2} (n/2)!}{n!}$.

[Hint: What does the number $\frac{n!}{2^{n/2} (n/2)!}$ remind you of?]

0.5. Hypergreen permutations

**Exercise 5.** Let $n \in \mathbb{N}$. We shall call a permutation $\pi \in S_n$ hypergreen if it satisfies both $\pi(1) < \pi(2)$ and $\pi^{-1}(1) < \pi^{-1}(2)$.

(a) Prove that any $\pi \in S_n$ satisfying $\pi(1) = 1$ must be hypergreen.
(b) Prove that the number of hypergreen permutations $\pi \in S_n$ that do not satisfy $\pi(1) = 1$ is $\left(\frac{n-2}{2}\right)^2 (n-4)!$.

[Hint: For (b), argue first that if $\pi \in S_n$ is hypergreen but does not satisfy $\pi(1) = 1$, then the four numbers 1, 2, $\pi(1)$, $\pi(2)$ are distinct.]

0.6. Counting the parts of all compositions

Recall that if $n \in \mathbb{N}$, then a composition of $n$ means a finite list $(a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = n$.

For example, the compositions of 3 are (3), (2, 1), (1, 2) and (1, 1, 1).

The length of a composition $(a_1, a_2, \ldots, a_k)$ of $n$ is defined to be $k$.

Exercise 6. Let $n$ be a positive integer. Prove that the sum of the lengths of all compositions of $n$ is $(n+1)2^{n-2}$. 