Exercise 1

Exercise 0.1. Let $D = (V, A, \psi)$ be an acyclic digraph. Then there is a list of elements $(v_1, v_2, \ldots, v_n)$ of $V$ such that each element of $V$ appears exactly once in the list $(v_1, v_2, \ldots, v_n)$, and whenever $i$ and $j$ are two elements of $[n]$, and $D$ features an arc which starts in $v_i$ and ends in $v_j$, then this implies that $i < j$.

Proof. Let $\text{Anc} : V \to \{	ext{subsets of } V\}$ be the function that maps each $v \in V$ to the set

$$\{w \in V \mid \text{ and there exists a walk from } w \text{ to } v\}.$$  

(As we know, the existence of a walk from $w$ to $v$ is equivalent to the existence of a path from $w$ to $v$; but we won’t actually need this.)

Since $V$ is a finite set, there exists some $n \in \mathbb{N}$ such that $|V| = n$. Consider this $n$. Since $|V| = n = |[n]|$, there exists a bijection $\phi : [n] \to V$. Fix this bijection $\phi$.

We now define the list $(v_1, v_2, \ldots, v_n)$ to be the list of all the $n$ elements $v \in V$ in increasing order of $|\text{Anc}(v)|$, where ties are broken as follows: If $v, w \in V$ satisfy $|\text{Anc}(v)| = |\text{Anc}(w)|$, then $v$ should be placed after $w$ if $\phi(v) > \phi(w)$ (and conversely, $w$ should be placed after $v$ if $\phi(w) > \phi(v)$).

We will now show that this list satisfies the two requirements in the claim. First of all, it is clear that each element of $V$ appears exactly once in this list, since this list has been constructed as a list of all elements of $V$ in some order.

It remains to check the second requirement. In other words, it remains to show that, if $i$ and $j$ are two elements of $[n]$, and if $D$ has an arc which starts at $v_i$ and ends at $v_j$, then $i < j$. 

Indeed, let $i$ and $j$ be two elements of $[n]$, and assume that $D$ has an arc which starts at $v_i$ and ends at $v_j$. We must prove $i < j$.

We will first show that $\text{Anc}(v_i) \subseteq \text{Anc}(v_j)$. Indeed, suppose that $w \in \text{Anc}(v_i)$. Thus, there exists a walk $\mathbf{a}$ from $w$ to $v_i$ (by the definition of $\text{Anc}(v_i)$). And because there is an arc which begins at $v_i$ and ends at $v_j$, then one can add that arc to the end of the walk $\mathbf{a}$ to construct a walk from $w$ to $v_j$. Hence, there exists a walk from $w$ to $v_j$, so $w \in \text{Anc}(v_j)$. So we have shown that $w \in \text{Anc}(v_i)$ implies that $w \in \text{Anc}(v_j)$. In other words, $\text{Anc}(v_i) \subseteq \text{Anc}(v_j)$.

We will next show that $v_j \not\in \text{Anc}(v_i)$. Suppose to the contrary that $v_j \in \text{Anc}(v_i)$. Then there exists some walk $\mathbf{b}$ from $v_j$ to $v_i$ (by the definition of $\text{Anc}(v_i)$). Because $D$ is acyclic, $\mathbf{b}$ must not contain any cycles, which means that $v_j$ does not appear in $\mathbf{b}$ except for at the very start. This means that the arc from $v_i$ to $v_j$ is not used in $\mathbf{b}$, as otherwise, $v_j$ would appear in the walk after that arc was used, which would by definition not be at the very start. Therefore, the walk constructed by adding that arc from $v_i$ to $v_j$ on to the end of $\mathbf{b}$ is a cycle in $D$ (going from $v_i$ to $v_j$). Thus, $D$ has a cycle. This contradicts the assumption that $D$ is acyclic. This contradiction reveals that $v_j \not\in \text{Anc}(v_i)$.

But there exists a walk from $v_j$ to $v_j$ (namely, the trivial walk $(v_j)$). Thus, $v_j \in \text{Anc}(v_j)$ (by the definition of $\text{Anc}(v_j)$). Contrasting this to $v_j \not\in \text{Anc}(v_i)$, we obtain $\text{Anc}(v_i) \neq \text{Anc}(v_j)$. Thus, $\text{Anc}(v_i)$ is a proper subset of $\text{Anc}(v_j)$. Hence, $|\text{Anc}(v_i)| < |\text{Anc}(v_j)|$.

Therefore, the vertex $v_i$ appears earlier than $v_j$ in the list $(v_1, v_2, \ldots, v_n)$ (due to how the list was constructed). In other words, $i < j$. This concludes our proof that the second requirement holds.

Hence, the constructed list satisfies the requirements of the claim. \hfill \Box

**Exercise 0.2.** Let $D$ be an acyclic multidigraph. A vertex $v$ of $D$ is said to be a sink if there is no arc of $D$ with source $v$.

If $u$ and $v$ are any two vertices of $D$, then:

- we write $u \rightarrow v$ if and only if $D$ has an arc with source $u$ and target $v$;

- we write $u \xrightarrow{*} v$ if and only if $D$ has a path from $u$ to $v$.

The so-called no-watershed condition says that for any three vertices $u$, $v$, and $w$ of $D$ satisfying $u \rightarrow v$ and $u \rightarrow w$, there exists a vertex $t$ of $D$ such that $v \xrightarrow{*} t$ and $w \xrightarrow{*} t$.

If the no-watershed condition holds, then for each vertex $p$ of $D$, there exists exactly one sink $q$ of $D$ such that $p \xrightarrow{*} q$.

**Proof.** Let $D$ be an acyclic multidigraph for which the no-watershed condition holds. Let $V$ be the vertex set of $D$, and let $h : V \rightarrow \mathbb{N}$ be the function that maps each $v \in V$ to the maximum length of a path in $D$ which begins at $v$.

We will first show that $h$ is well defined. Observe that $D$ has finitely many vertices. Also, each path in $D$ goes through each vertex of $D$ at most one time. Since the length of a path is equal to the number of edges taken in that path, which is equal to the number of vertices taken in that path minus one, the length of a path in $D$ must be an integer $\leq |V| - 1$. So the set of lengths of paths which begin at a vertex $v \in V$ is some subset of $\{0, 1, \ldots, |V| - 1\}$. And since this subset is a finite nonempty set of integers (nonempty because the trivial path $(v)$ always exists), it must have a maximum value. Hence, for all vertices $v \in V$, the number $h(v)$ is defined.
We will next show that if \( u, v \in V \), and if there exists a path of nonzero length from \( u \) to \( v \), then
\[
h(v) < h(u).
\] (1)

[Proof of (1). Let \( u, v \in V \), and suppose that there exists a path of nonzero length from \( u \) to \( v \). Consider such a path \( a \); thus, its length is positive. By the definition of \( h(v) \), we know that there is a path \( b \) of length \( h(v) \) which begins at \( v \). Now consider the walk \( c \) formed by adding the path \( b \) onto the end of the path \( a \). This new walk \( c \) is still a path (since otherwise, it would contain a cycle, but this would contradict the acyclicity of \( D \)), and has length \( > h(v) \) (indeed, its length equals the sum of the lengths of \( a \) and \( b \), but the former length is positive and the latter is \( h(v) \)). Thus, \( c \) is a path in \( D \) which begins at \( u \) and has length \( > h(v) \). Since \( h(u) \) is the maximum length of a path in \( D \) which starts at \( u \), we thus conclude that \( h(u) \) is at least as large as the length of \( c \), which is \( > h(v) \). Hence, \( h(u) > h(v) \). This proves (1).]

From (1), we immediately obtain the following: If \( u, v \in V \), and if there exists a path from \( u \) to \( v \), then
\[
h(v) \leq h(u).
\] (2)

(Indeed, if this path has nonzero length, then this inequality follows from (1), whereas otherwise it follows from \( v = u \).)

The exercise claims that for each vertex \( v \) of \( D \), there exists exactly one sink \( q \) of \( D \) such that there is a path from \( v \) to \( q \). We will now prove this claim by strong induction on \( h(v) \).

For the base case, suppose that \( v \in V \) and \( h(v) = 0 \). Since \( h(v) = 0 \), there are no paths in \( D \) of nonzero length which start at \( v \). This is possible only if there are no arcs in \( D \) which begin at \( v \), which implies that \( v \) is a sink. Thus, \( v \) is a sink; hence, there exists only one vertex \( q \in V \) for which there exists a path from \( v \) to \( q \) (namely, \( v \) itself). And since \( v \) is a sink, this means that there exists a path from \( v \) to exactly one sink (itself), and no other sinks (or even vertices for that matter). This completes the induction base.

Now, to the induction step. Let \( n \in \mathbb{N} \). Assume that the claim holds for all vertices \( u \in V \) such that \( h(u) < n \). Consider a vertex \( v \in V \) such that \( h(v) = n \). We need to prove the claim for this vertex \( v \). If \( h(v) = 0 \), then this has already been proven in the above induction base; thus, we assume that \( h(v) > 0 \). Hence, there exists a path of nonzero length which originates at \( v \). Thus, there exists an arc with source \( v \).

Let \( B \) be the set of the targets of all arcs with source \( v \). Since such arcs do exist (as we have just seen), we have \( B \neq \emptyset \). And also, each path which begins at \( v \) must have its second vertex be a vertex in \( B \). And further, if \( w \in B \), then there exists a path from \( v \) to \( w \), so that \( h(w) < h(v) \) (by (1)).

Now let \( w_1 \in B \). Since \( B \neq \emptyset \), such a \( w_1 \) must exist. And since \( h(w_1) < h(v) = n \), we can apply the induction hypothesis to \( w_1 \) instead of \( v \). We conclude that there exists exactly one sink \( x \in V \) such that there is a path from \( w_1 \) to \( x \). Consider this \( x \). Since \( w_1 \in B \), there exists a path from \( v \) to \( w_1 \), so there exists a path from \( v \) to \( x \) (via \( w_1 \)).

Now let \( w_2 \in B \) be arbitrary (in particular, \( w_2 \) may be equal to \( w_1 \)). Since \( w_1, w_2 \in B \), we have \( v \rightarrow w_1 \) and \( v \rightarrow w_2 \). Since the no-watershed condition holds, we conclude that there exists a vertex \( t \in V \) such that there is a path from \( w_1 \) to \( t \) and there is a path from \( w_2 \) to \( t \). Therefore, using (2), we obtain \( h(t) \leq h(w_1) < h(v) = n \). So by the induction hypothesis (applied to \( t \) instead of \( v \)), there exists exactly one sink \( y \in V \) such that there is a path from \( t \) to \( y \). Consider this \( t \). Concatenating a path from \( w_2 \) to \( t \) with a path from \( t \) to \( y \), we obtain a walk from \( w_2 \) to \( y \), thus a path from \( w_2 \) to \( y \). Similarly, we find that there is a path from \( w_1 \) to \( y \).

So \( y \) is a sink for which there exists a path from \( w_1 \) to \( y \). But we have previously defined \( x \) to be the only such sink. Therefore, \( y = x \). But recall that there is a path from \( w_2 \) to \( y \).
In other words, there is a path from \( w_2 \) to \( x \) (since \( y = x \)).

We thus have shown that for each \( w_2 \in B \), there is a path from \( w_2 \) to \( x \).

Now, consider any sink \( z \) for which there is a path from \( v \) to \( z \). This path has nonzero length (since \( h(v) > 0 \), so that \( v \) itself is not a sink), and thus has a second vertex. Denote this second vertex by \( w_2 \); thus, \( w_2 \in B \), so that (as we have just seen) there is a path from \( w_2 \) to \( x \). Also, from \( w_2 \in B \), we obtain \( h(w_2) < h(v) = n \), so that we can apply the induction hypothesis to \( w_2 \) instead of \( v \). We thus conclude that there is exactly one sink \( q \) such that there is a path from \( w_2 \) to \( q \). This proves the claim for our vertex \( v \). So the induction step is complete, and the claim of the exercise follows.

**EXERCISE 4**

**PART A**

**Definition 0.3.** Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \). The graph \( K_{n,m} \) is defined to be the simple graph with \( n+m \) vertices

\[ 1, 2, \ldots, n, -1, -2, \ldots, -m \]

and \( nm \) edges

\[ \{i, -j\} \quad \text{for all} \ i \in [n] \ \text{and} \ j \in [m]. \]

(Note that \( (K_{n,m}; \{1, 2, \ldots, n\}; \{-1, -2, \ldots, -m\}) \) is a bipartite graph, called the complete bipartite graph.)

**Exercise 0.4.** Let \( m, n \in \mathbb{N} \). Then the chromatic polynomial of \( K_{n,m} \) is given by

\[ \chi_{K_{n,m}} = \sum_{i=0}^{n} \text{sur}(n,i) \binom{x}{i} (x-i)^m \]

*Proof.* Refer to the vertices \( 1, 2, \ldots, n \) of \( K_{n,m} \) as the *positive vertices* of \( K_{n,m} \), and to the vertices \( -1, -2, \ldots, -m \) as the *negative vertices* of \( K_{n,m} \).

Observe that for any color used in a proper coloring of \( K_{n,m} \), that color can not appear on both a positive vertex and a negative vertex, since there is an edge connecting each positive vertex to each negative vertex. Hence, in a proper coloring of \( K_{n,m} \), the set of colors used to color the positive vertices, and the set of colors used to color the negative vertices are disjoint.

Now, let \( k \in \mathbb{N} \). Recall that the value \( \chi_{K_{n,m}}(k) \) of the chromatic polynomial is equal to the number of proper \( k \)-colorings of \( K_{n,m} \). We will count these \( k \)-colorings now. Let \( C = [k] \); thus, a \( k \)-coloring of \( K_{n,m} \) is a map from the set of vertices of \( K_{n,m} \) to \( C \). We can construct such a coloring \( f \) in the following four steps:

- Choose the number \( i \) of colors that will be used to color the positive vertices (so \( i \) will be \( |f([n])| \)). This is a number between 0 and \( n \).
- Choose the set \( C_p \) of colors that will be used to color the positive vertices. This must be an \( i \)-element subset of the \( k \)-element set \( C \). Thus, there are \( \binom{k}{i} \) options here.
• Color the positive vertices with the colors from $C_p$, using each color at least once. This is tantamount to choosing a surjective map from the $n$-element set $[n]$ to the $i$-element set $C_p$ (sending each positive vertex to its color); thus, there are $\operatorname{sur}(n,i)$ options for it.

• Finally, color the negative vertices. Their colors need to be chosen from the $k-i$ colors that don’t belong to $C_p$ (since the set of colors used to color the positive vertices, and the set of colors used to color the negative vertices must be disjoint in a proper $k$-coloring), but we don’t have to use each color. Hence, this is tantamount to choosing a map from the $m$-element set $\{-1,-2,\ldots,-m\}$ to the $k-i$-element set $C\setminus C_p$. Thus, there are $(k-i)^m$ options at this step.

At the end of this algorithm, all vertices of $K_{n,m}$ are colored, and the resulting $k$-coloring is proper (because each edge connects a positive vertex with a negative vertex, and we’ve ensured that the latter vertex has a different color than the former). Hence, the number of all proper $k$-colorings of $K_{n,m}$ is $\sum_{i=0}^{n} \binom{k}{i} \operatorname{sur}(n,i) (k-i)^m$ (which we get by multiplying the numbers of options in the above algorithm). On the other hand, this is $\chi_{K_{n,m}}(k)$ (as we already showed). Comparing the two results, we find

$$\chi_{K_{n,m}}(k) = \sum_{i=0}^{n} \binom{k}{i} \operatorname{sur}(n,i) (k-i)^m.$$

Now we have proven this for each $k \in \mathbb{N}$. Thus, the two polynomials

$$\chi_{K_{n,m}}(x) \quad \text{and} \quad \sum_{i=0}^{n} \binom{x}{i} \operatorname{sur}(n,i) (x-i)^m$$

are equal to each other on each point $k \in \mathbb{N}$. This means that they are equal to each other on infinitely many points. Hence, they must be identical as polynomials (by the “polynomial identity trick”). In other words,

$$\chi_{K_{n,m}}(x) = \sum_{i=0}^{n} \binom{x}{i} \operatorname{sur}(n,i) (x-i)^m = \sum_{i=0}^{n} \operatorname{sur}(n,i) \binom{x}{i} (x-i)^m.$$

\[\square\]

**Part B**

**Exercise 0.5.** For all $m, n \in \mathbb{N}$, it holds that

$$\sum_{i=0}^{n} \operatorname{sur}(n,i) \binom{x}{i} (x-i)^m = \sum_{i=0}^{m} \operatorname{sur}(m,i) \binom{x}{i} (x-i)^n.$$

**Proof.** Let $m, n \in \mathbb{N}$. We claim that the graphs $K_{n,m}$ and $K_{m,n}$ are identical up to the names of their vertices.\(^1\)

Indeed, the graph $K_{n,m}$ has vertices $1, 2, \ldots, n$ and $-1, -2, \ldots, -m$, with edges $\{i, -j\}$ for $i \in [n]$ and $j \in [m]$. If one renames each vertex $k$ as $-k$, and updates the formula for edges such that it is consistent with the new names, then the resulting graph has the

\(^1\)That is, we can rename the vertices of $K_{n,m}$ in such a way that the resulting graph is $K_{m,n}$. In more rigorous language, we are saying that the graphs $K_{n,m}$ and $K_{m,n}$ are isomorphic.

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vertices $-1, -2, \ldots, -n$ and $1, 2, \ldots, m$, with edges $\{-i, j\}$ for $i \in [n]$ and $j \in [m]$. But this is precisely the graph $K_{m,n}$. Hence, $K_{n,m}$ is equal to the graph $K_{m,n}$, except for the fact that the vertices are named differently.

And since the way the vertices of a graph are named does not in any way affect the number of proper colorings of a graph, it follows that $\chi_{K_{n,m}}(k) = \chi_{K_{m,n}}(k)$ for each $k \in \mathbb{N}$. In other words, the polynomials $\chi_{K_{n,m}}$ and $\chi_{K_{m,n}}$ are equal to each other on each point $k \in \mathbb{N}$. Hence, $\chi_{K_{n,m}} = \chi_{K_{m,n}}$.

In part (a), it was shown that $\chi_{K_{n,m}} = \sum_{i=0}^{n} \text{sur}(n, i) \left(\binom{x}{i}\right) (x - i)^m$. And swapping $m$ and $n$ in this formula yields $\chi_{K_{m,n}} = \sum_{i=0}^{m} \text{sur}(m, i) \left(\binom{x}{i}\right) (x - i)^n$. Thus, the equality $\chi_{K_{n,m}} = \chi_{K_{m,n}}$ rewrites as $\sum_{i=0}^{n} \text{sur}(n, i) \left(\binom{x}{i}\right) (x - i)^m = \sum_{i=0}^{m} \text{sur}(m, i) \left(\binom{x}{i}\right) (x - i)^n$. \qed