Subconvexity bounds in depth-aspect for automorphic L-functions on GL_2

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From a spectral identity we obtain asymptotics with error-term for the second integral moments of families of automorphic L-functions for GL_2 over fields twisted by idele characters with ramification at a fixed finite place. The power-saving in the error term breaks convexity at this non-archimedean place.

[1] 1.1 INTRODUCTION

The *convexity* or *trivial bound* for the zeta function is

$$|\zeta(\frac{1}{2}+it)| \ll |t|^{\frac{1}{4}+\epsilon}$$

Any improvement over $\frac{1}{4}$ in this upper bound "breaks convexity". Various authors have obtained subconvexity bounds in different aspects. [Weyl 1921] gave a subconvex bound

$$|\zeta(\frac{1}{2}+it)| \ll |t|^{\frac{1}{6}+\epsilon}$$

[Burgess 1962] broke convexity in conductor aspect for Dirichlet L-functions over $\mathbb{Q}$. Subconvexity bounds were obtained for GL_2 L-functions in [Good 1982, 1986], [Meurman 1987] and [Duke-Friedlander-Iwaniec 1993, 1994, 2001]. In recent years, subconvexity results were obtained by several authors including Kowalski, Michel, Vanderkam and Venkatesh (see [Kowalski-Michel-Vanderkam 2002] and [Michel-Venkatesh 2006]).

Until recently, all these results concerned automorphic L-functions over $\mathbb{Q}$, or over quadratic extensions of $\mathbb{Q}$, not over number fields. In 2006, [Diaconu-Goldfeld 2006a, 2006b] reconsidered the cases of groundfield $\mathbb{Q}$ and complex quadratic extensions. Then [Diaconu-Garrett 2008] obtained asymptotics with error-term for second integral moments of GL_2 automorphic L-functions over number fields, by a spectral identity with power-saving in the error term for averages not only over the critical line but also over twists by grössencharakters:

$$\sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \, M_\chi(t)\, dt$$

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where $M_\chi(t)$ are smooth weights. They showed that this breaks t-aspect convexity.

In a recent preprint, Michel and Venkatesh have announced a joint convexity bound in all aspects using ergodic theory and spectral theory.

Here we take Diaconu-Garrett's ideas in a different direction. Fixing a GL_2 automorphic L-function over a number field, we deform the data at a fixed *non-archimedean* place v_1, and allow χ to have arbitrary ramification at v_1.

2.1 THE MAIN RESULT

We break convexity in the χ-depth-aspect for a family of L-functions $L(\frac{1}{2}+it, f\otimes\chi)$, where χ has arbitrary ramification at a fixed finite prime v_1. For a cuspform f on $GL_2(k)$, where k is of degree d over $\mathbb{Q}$, the χ-depth-aspect convexity bound is

$$L(\frac{1}{2}+it, f\otimes\chi) \ll q^{N(\frac{d}{2}+\epsilon)}$$

where q^N, with $N \geq 1$, is the conductor of χ. We prove

$$L(\frac{1}{2}+it, f\otimes\chi) \ll (q^N)^{\frac{d-1+\vartheta}{2}+\epsilon} \quad (\text{some } \vartheta < 1)$$

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3.1 THE MOMENT EXPANSION

The integral moment expansion is obtained by unwinding an integral representation

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} P\acute{e}\cdot|f|^2$$

where $P\acute{e}$ is a Poincaré series and f is a cuspform on GL_2. We use ideas from section 2 in [Diaconu-Garrett 2008] to reformulate the Poincaré series as a single object. The moment expansion is a sum of weighted integrals of L-functions $L(s, f\otimes\chi)$ of twists of f by characters χ. The weight functions depend on archimedean data and data associated with the finite place v_1 where χ has ramification. We will then obtain asymptotics from the weight functions.

3.1.1 Unwinding to an Euler Product

Define subgroups of $G = GL_2$:

$$P = \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\right\},\ N = \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right\},\ H = \left\{\left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\},\ Z = \text{ center of } G,\ M = ZH = \left\{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\right\}$$

For any place v of k, let $K_v^{\max}$ be the standard maximal compact subgroup: For finite v,

$$K_v^{\max} = GL_2(\mathfrak{o}_v)$$

and for infinite v,

$$K_v^{\max} = \begin{cases} O_2 & (v \approx \mathbb{R}) \\ U_2 & (v \approx \mathbb{C}) \end{cases}$$

The Poincaré series $P\acute{e}$ is of the form

$$P\acute{e}(g) = \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g) \qquad (\text{where } g\in G_{\mathbb{A}})$$

for suitable functions φ on $G_{\mathbb{A}}$ defined as follows. Let

$$\varphi = \otimes_v \varphi_v$$

where for finite primes $v \neq v_1$,

$$\varphi_v(g) = \begin{cases} \chi_{0,v}(m) = \left|\frac{a}{d}\right|_v^{s'} & (\text{for } g = mk,\ m = \left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right) \in M,\ s'\in\mathbb{C},\ k\in K_v^{\max}) \\ 0 & (\text{otherwise})\end{cases}$$

For finite $v = v_1$ (at which χ is allowed to be ramified)

$$\varphi_v(mg) = \left|\frac{a}{d}\right|_v^{s'} \cdot \varphi_v(g) \quad (m\in M_v,\ g\in G_v)$$

The data determining φ_v for $v = v_1$ consists of its values on N_v where our simple choice is

$$\varphi_v\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right) = \begin{cases} 1 & (\text{for } x\in\mathfrak{o}_v) \\ |x|_v^{-w'} & (\text{for } w'\in\mathbb{C},\ x\notin\mathfrak{o}_v)\end{cases}$$

For infinite v require right K_v-invariance and left equivariance:

$$\varphi_v(mg) = \left|\frac{a}{d}\right|_v^{s'} \cdot \varphi_v(g) \quad (m\in M_v,\ g\in G_v)$$

where

$$\varphi_v\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right) = \begin{cases} (1+|x|^2)^{-\frac{w}{2}} & (\text{for } v\approx\mathbb{R},\ w\in\mathbb{C}) \\ (1+x\overline{x})^{-w} & (\text{for } v\approx\mathbb{C})\end{cases}$$

The Poincaré series $P\acute{e}$ converges absolutely and locally uniformly for $\Re(s') > 1$, $\Re(w) > 1$ for all $v|\infty$, and for $\Re(w') > 1$, by Proposition 2.6 in [Diaconu-Garrett 2008].

We want to show that

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} P\acute{e}\cdot |f|^2\, dg$$

is an integral of products of local factors of standard L-functions. First, the Fourier expansion of a cuspform f on $G_{\mathbb{A}}$ is

$$f(g) = \sum_{\xi\in Z_k\backslash M_k} W_f(\xi g)$$

where W_f is the Whittaker function of f and $W_f = \otimes_v W_{f,v}$ is the factorization of W_f into local data. So

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} P\acute{e}\cdot|f|^2\,dg \;=\; \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g)\,|f(g)|^2\,dg \;=\; \int_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}} \varphi(g)\,|f(g)|^2\,dg$$

$$= \int_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}} \varphi(g) \sum_{\xi\in Z_k\backslash M_k} W_f(\xi g)\,\overline{f}(g)\,dg \;=\; \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,W_f(g)\,\overline{f}(g)\,dg$$

Let C be the idele class group $GL_1(k)\backslash GL_1(\mathbb{A})$ and $\hat{C}$ its dual. $\hat{C}\approx\mathbb{R}\times\hat{C}_0$ where $\hat{C}_0$ is discrete. The Mellin transform and inversion are

$$\begin{aligned} f(x) &= \int_{\hat{C}} \int_C f(y)\,\chi^{-1}(y)\,dy\,\chi(x)\,d\chi \\ &= \sum_{\chi'\in\hat{C}_0} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} \int_C f(y)\,{\chi'}^{-1}(y)\,|y|^{-s}\,dy\,\chi'(x)\,|x|^s\,ds \end{aligned}$$

With $Z_{\mathbb{A}}M_k\backslash M_{\mathbb{A}}\approx C$, and for finite $v\neq v_1$,

$$\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,W_f(g)\,\overline{f}(g)\,dg \;=\; \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,W_f(g)\Big(\int_{\hat{C}}\int_{Z_{\mathbb{A}}M_k\backslash M_{\mathbb{A}}} \overline{f}(m'g)\,\chi(m')\,dm'\,d\chi\Big)\,dg$$

$$= \int_{\hat{C}} \Big(\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,W_f(g) \int_{Z_{\mathbb{A}}M_k\backslash M_{\mathbb{A}}} \sum_{\xi\in Z_k\backslash M_k} \overline{W}_f(\xi m'g)\,\chi(m')\,dm'\,dg\Big)\,d\chi$$

$$= \int_{\hat{C}} \Big(\int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\,W_f(g) \int_{Z_{\mathbb{A}}\backslash M_{\mathbb{A}}} \overline{W}_f(m'g)\,\chi(m')\,dm'\,dg\Big)\,d\chi$$

$$= \int_{\hat{C}} \prod_v \Big(\int_{Z_v\backslash G_v} \int_{Z_v\backslash M_v} \varphi_v(g_v)\,W_{f,v}(g_v)\,\overline{W}_{f,v}(m'_v g_v)\,\chi_v(m'_v)\,dm'_v\,dg_v\Big)\,d\chi$$

Suppress finite $v\neq v_1$ and write the v^{th} local integral as

$$\int_{Z\backslash G}\int_{Z\backslash M} \varphi(g)\,W_f(g)\,\overline{W}_f(m'g)\,\chi(m')\,dm'\,dg$$

Invoke the v-adic Iwasawa decomposition $G=MNK$ and rewrite the integral as

$$\int_{Z\backslash MNK}\int_{Z\backslash M} \varphi(mnk)\,W_f(mnk)\,\overline{W}_f(m'mnk)\,\chi(m')\,dm'\,dm\,dn\,dk$$

For simplicity, take φ and f to be right $K_v^{\max}$-invariant for finite $v\neq v_1$. This gives

$$\int_{Z\backslash MN}\int_{Z\backslash M} \varphi(mn)\,W_f(mn)\,\overline{W}_f(m'mn)\,\chi(m')\,dm'\,dm\,dn$$

Replace m' by $m'm^{-1}$ to get

$$\int_{Z\backslash MN}\int_{Z\backslash M} \varphi(mn)\,W_f(mn)\,\overline{W}_f(m'n)\,\chi(m')\,\chi^{-1}(m)\,dm'\,dm\,dn$$

The Whittaker function has the equivariance

$$W_f(ng) = \psi(n)\,W_f(g)\quad (n\in N_{\mathbb{A}})$$

Thus,

$$W_f(mn) = W_f(mnm^{-1}m) = \psi(mnm^{-1})\, W_f(m) \quad (\text{since } mnm^{-1} \in N)$$

and

$$\overline{W}_f(m'n) = \overline{W}_f(m'nm^{-1}m) = \overline{\psi}(m'nm'^{-1})\, \overline{W}_f(m')$$

so obtaining

$$\int_{Z\backslash MN} \int_{Z\backslash M} \varphi(mn)\, W_f(m)\, \overline{W}_f(m')\, \chi(m')\, \chi^{-1}(m)\, \psi(mnm^{-1})\, \overline{\psi}(m'nm'^{-1})\, dm'\, dm\, dn$$

Let

$$X(m, m') = \int_N \varphi(n)\, \psi(mnm^{-1})\, \overline{\psi}(m'nm'^{-1})\, dn$$

We get

$$\int_{Z\backslash M} \int_{Z\backslash M} \chi_0(m)\, W_f(m)\, \overline{W}_f(m')\, \chi(m')\, \chi^{-1}(m)\, X(m, m')\, dm'\, dm$$

Now

$$W_f(mn) = \psi(mnm^{-1}) \cdot W_f(m)$$

and

$$W_f(mn) = W_f(m) \cdot 1$$

by the right K-invariance of W_f. So for $W_f(m) \neq 0$, $\psi(mnm^{-1}) = 1$, and $X(m, m') = 1$ for m, m' in the support of W_f. So

$$\int_{Z\backslash M} (\chi_0 \cdot \chi^{-1})(m)\, W_f(m)\, dm \;\cdot\; \int_{Z\backslash M} \chi(m')\, \overline{W}_f(m')\, dm'$$
$$= L_v(\chi_{0,v} \cdot \chi_v^{-1}\, |y|_v^{\frac{1}{2}},\, f) \cdot L_v(\chi_v\, |y'|_v^{\frac{1}{2}},\, \overline{f}) \quad (\text{where} \quad m = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right),\;\; m' = \left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right))$$

is a product of local factors at finite primes $v \neq v_1$. Thus the integral can be written as

$$I(\chi_0) = \sum_{\chi \in \hat{C}_0} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L(\chi_0{\cdot}\chi^{-1}\, |y|^{1-s},\, f) \cdot L(\chi\, |y'|^s,\, \overline{f}) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, \chi_0, \chi)\, ds$$

where

$$\mathcal{K}_\infty(s, \chi_0, \chi) = \prod_{v|\infty} \mathcal{K}_v(s, \chi_{0,v}, \chi_v)$$

and

$$\mathcal{K}_v(s, \chi_{0,v}, \chi_v) \;=\; \int_{Z_v\backslash M_v N_v} \int_{Z_v\backslash M_v} \varphi_v(m_v n_v)\, W_{f,v}(m_v n_v)\, \overline{W}_{f,v}(m'_v n_v) \cdot$$
$$\chi_v(m'_v)\, |m'_v|_v^{s-\frac{1}{2}}\, \chi_v^{-1}(m_v)\, |m_v|_v^{\frac{1}{2}-s}\, dm'_v\, dm_v\, dn_v$$

$\mathcal{K}_{v_1}(w', \chi_{v_1}) =$

$$\int_{k_v^\times} \int_{k_v^\times} \chi(y)\, |y|_{v_1}^s\, \chi^{-1}(y')\, |y'|_{v_1}^{1-s}\, W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \overline{W}\left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \int_{k_v} \overline{\psi}(x \cdot (y - y'))\, \varphi_{v_1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) dx\ dy\, dy'$$

where χ is ramified at $v = v_1$. The non-decoupled integrals $\mathcal{K}_v(s, \chi_{0,v}, \chi_v)$ and $\mathcal{K}_{v_1}(w', \chi_{v_1})$, which represent the weight functions, will be subsequently computed.

The Poincaré series $P\acute{e}$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing $s' = 0$ and $w' = 1$. As a function of w', for $s' = 0$, it is holomorphic in the half-plane $\Re(w') > \frac{11}{18}$ ([Kim-Shahidi 2002], [Kim 2005]), except for $w' = 1$ where it has a pole of order 1. This can be seen from the spectral decomposition of $P\acute{e}$ (in Section 4), and the proof of Theorem 4.17 in [Diaconu-Garrett 2008].

For $\Re(s')$ and $\Re(w')$ sufficiently large, the integral $I(\chi_0) = I(s', w')$ is

$$I(s', w') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L(\chi^{-1}\, |.|^{s'+1-s},\ f) \cdot L(\chi\, |.|^s,\ \overline{f}) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, s', w, \chi)\, ds$$

where S is a finite set of places including archimedean places, and the sum is over the set $\hat{C}_{0,S}$ of characters ramified at v_1. $I(s', w')$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing the point $s' = 0$, $w' = 1$, and $I(0, w')$ is holomorphic for $\Re(w') > \frac{11}{18}$ except for $w' = 1$ where it has a pole of order 1.

We will find asymptotics for $\mathcal{K}_{v_1}(w', \chi_{v_1})$ and $\mathcal{K}_\infty(s, s', w, \chi)$ in Section 3.1.4, shift the line of integration to $\Re(s) = \frac{1}{2}$ and set $s' = 0$. Thus for $\Re(w')$ sufficiently large

$$\begin{aligned} I(0, w') &= \sum_{\chi \in \hat{C}_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(\chi^{-1}\, |.|^{\frac{1}{2} - it},\ f)\ \cdot\ L(\chi\, |.|^{\frac{1}{2} + it},\ \overline{f})\ \cdot\ \mathcal{K}_{v_1}(w', \chi_{v_1})\ \cdot\ \mathcal{K}_\infty(\frac{1}{2} + it, 0, w, \chi)\, dt \\ &= \sum_{\chi} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it,\ f \otimes \chi)|^2\ \cdot\ \mathcal{K}_{v_1}(w', \chi_{v_1})\ \cdot\ \mathcal{K}_\infty(\frac{1}{2} + it, 0, w, \chi)\, dt \end{aligned}$$

3.1.2 The non-decoupled integrals

The nondecoupled integral $\mathcal{K}_{v_1}(w', \chi_{v_1})$ is

$$\int_{k_v^\times} \int_{k_v^\times} \chi(y)\, |y|_v^s\, \chi^{-1}(y')\, |y'|_v^{1-s}\, W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \overline{W}\left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \int_{k_v} \overline{\psi}(x \cdot (y - y'))\, \varphi_v\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) dx\ dy\, dy'$$

where there is ramification of χ at finite $v = v_1$. Henceforth, suppress the index v. ψ is the standard additive character trivial on the local integers $\mathfrak{o}$ and nontrivial on $\varpi^{-1}\mathfrak{o}$. χ is a ramified multiplicative character. W is a spherical Whittaker function of the form

$$W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) = \begin{cases} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $ord(y) = n$. Compute the integral in y and y', and then compute the integral in x. Now

$$\overline{\psi}(x(y-y')) = \overline{\psi}(xy - xy') = \overline{\psi}(xy) \cdot \psi(xy')$$

Thus the integrals in y and y' are:

$$\int_{k^\times} \overline{\psi}(xy)\,\chi(y)\,|y|^s\ W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy \cdot \int_{k^\times} \psi(xy')\,\chi^{-1}(y')\,|y'|^{1-s}\,\overline{W}\left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right) dy'$$

In the integral in y, replace y with $y\eta$ with $\eta \in \mathfrak{o}^\times$:

$$\int_{k^\times} \Big(\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\,\chi(y\eta)\,d\eta\Big)\,|y|^s\ W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy$$

Consider the inner integral:

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\,\chi(y\eta)\,d\eta$$

Let N be the ord of the conductor of ramified χ. So χ is trivial on the subgroup $1 + \mathfrak{m}^N$ of $\mathfrak{o}^\times$ and non-trivial on $1 + \mathfrak{m}^{N-1}$.

As in [Weil 1974], a standard computation shows that

$$\int_{k^\times} \psi(xy)\,\chi(y)\,dy \ = \ 0 \ \text{ unless } \ ord(x) = -N$$

So we claim that our inner integral is zero unless $ord(xy) = -N$. So $ord(y) = -ord(x) - N$. The integral

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\,\chi(y\eta)\,d\eta$$

is a Gauss sum

$$\mathfrak{g}(\chi, \psi) = \int_{\mathfrak{o}^\times} \chi(x) \cdot \overline{\psi}(\frac{x}{\varpi^N})\,dx = \frac{q^{2-N}}{(q-1)^2}$$

The integral in y' is:

$$\int_{k^\times} \Big(\int_{\mathfrak{o}^\times} \psi(xy't)\,\overline{\chi}(y't)\,dt\Big)\,|y'|^{1-s}\ \overline{W}\left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right) dy', \qquad t \in \mathfrak{o}^\times$$

the conjugate of the integral in y. Replacing $y\eta$ with u, and x with m, the integrals over $\mathfrak{o}^\times$ in y and y' are

$$\Big|\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\,\chi(y\eta)\,d\eta\Big|^2 \ = \ \Big|\int_{\mathfrak{o}^\times} \overline{\psi}(\frac{mu}{\varpi^N})\,\chi(u)\,du\Big|^2 \ = \ \frac{q^{2-N}}{(q-1)^2}$$

The entire nondecoupled local integral becomes

$$\frac{q^{2-N}}{(q-1)^2}\Big[\int_{k^\times} |y|^s\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy \cdot \int_{k^\times} |y'|^{1-s}\ \overline{W}\left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right) dy' \cdot \int_k \varphi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) dx\Big]$$

The integral is zero unless $ord(y) = -ord(x) - N$, so rewrite the integral as

$$\frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot \int_{ord(y)=-ord(x)-N} |y|^s\, W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy \cdot \int_{ord(y')=-ord(x)-N} |y'|^{1-s}\, \overline{W}\left(\begin{smallmatrix} y' & 0 \\ 0 & 1 \end{smallmatrix}\right) dy'\ dx$$

$$= \frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot \int_{ord(y)=-ord(x)-N} |y|\, |W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)|^2\, dy\, dx$$

Since $ord(y) = -ord(x) - N$, y can be written as $y = \frac{t}{\varpi^N x}$ with $t \in \mathfrak{o}^\times$. So the entire integral is

$$\frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot |\frac{1}{\varpi^N x}|\, |W\left(\begin{smallmatrix} \frac{1}{\varpi^N x} & 0 \\ 0 & 1 \end{smallmatrix}\right)|^2\, dx$$

Now $W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is supported on $\mathfrak{o} \cap k^\times$, so $ord(x) \le -N$. Then $x \notin \mathfrak{o}$. Thus, we integrate over $k^\times$, and the integral becomes

$$\frac{q^{2-N}}{(q-1)^2} \cdot \int_k |x|^{-w'} \cdot |\frac{1}{\varpi^N x}|\, |W\left(\begin{smallmatrix} \frac{1}{\varpi^N x} & 0 \\ 0 & 1 \end{smallmatrix}\right)|^2\, dx$$

$$= \frac{q^{1-N}}{q-1} \cdot \int_{k^\times} |x|^{1-w'} \cdot |\frac{1}{\varpi^N x}|\, |W\left(\begin{smallmatrix} \frac{1}{\varpi^N x} & 0 \\ 0 & 1 \end{smallmatrix}\right)|^2\, dx$$

Invert x to get

$$\frac{q^{1-N}}{q-1} \cdot \int_{k^\times} |x|^{w'-1} \cdot |\frac{x}{\varpi^N}|\, |W\left(\begin{smallmatrix} \frac{x}{\varpi^N} & 0 \\ 0 & 1 \end{smallmatrix}\right)|^2\, dx$$

Replace x by $\varpi^N x$ and let $ord(x) = \ell$ to get

$$\mathcal{K}_{v_1}(w', \chi_{v_1}) = \frac{q^{1-N}}{q-1} \cdot q^{-Nw'} \cdot q^N \cdot \int_{k^\times} |x|^{w'-1} \cdot |x| \cdot |W\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)|^2\, dx$$

$$= \frac{q}{q-1} \cdot q^{-Nw'} \cdot \sum_{\ell=0}^{\infty} q^{-\ell w'} \cdot \frac{\alpha^{\ell+1} - \beta^{\ell+1}}{\alpha - \beta} \cdot \frac{\overline{\alpha}^{\ell+1} - \overline{\beta}^{\ell+1}}{\overline{\alpha} - \overline{\beta}}$$

$$= \frac{q^{1-Nw'}}{q-1} \cdot \frac{1 - |\alpha|^2|\beta|^2 q^{-2w'}}{(1-|\alpha|^2 q^{-w'})(1-|\beta|^2 q^{-w'})(1-\overline{\alpha}\beta q^{-w'})(1-\alpha\overline{\beta} q^{-w'})}$$

3.1.3 Bound

Now the latter expression, apart from the factor of $q^{-Nw'}$, is independent of the conductor q^N of χ, so

$$\mathcal{K}_{v_1}(w', \chi_{v_1}) \ll (q^N)^{-w'}$$

4.1 SPECTRAL DECOMPOSITION OF THE POINCARÉ SERIES

4.1.1 The Cuspidal Part

Let F be a cuspform on $G_{\mathbb{A}}$ generating a spherical representation locally everywhere, and suppose F is a spherical vector everywhere locally. The F^{th} (cuspidal) component of the spectral decomposition of the Poincaré series is $\dfrac{\langle Pé, F\rangle \cdot F}{\langle F, F\rangle}$. Compute

$$\langle Pé, F\rangle \;=\; \int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} Pé(g)\cdot \overline{F}(g)\, dg \;=\; \int_{Z_{\mathbb{A}} G_k\backslash G_{\mathbb{A}}} \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g)\, \overline{F}(g)\, dg$$

$$= \int_{Z_{\mathbb{A}} M_k\backslash G_{\mathbb{A}}} \varphi(g)\, \overline{F}(g)\, dg \;=\; \int_{Z_{\mathbb{A}} M_k\backslash G_{\mathbb{A}}} \varphi(g) \sum_{\xi\in Z_k\backslash M_k} \overline{W}_F(\xi g)\, dg \;=\; \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\, \overline{W}_F(g)\, dg$$

$$= \prod_{v<\infty,\, v\neq v_1} \int_{Z_v\backslash G_v} \varphi_v(g)\, \overline{W}_{F,v}(g)\, dg \cdot \int_{Z_{v_1}\backslash G_{v_1}} \varphi_{v_1}(g)\, \overline{W}_{F,v_1}(g)\, dg \cdot$$

$$\prod_{v|\infty} \int_{Z_v\backslash G_v} \varphi_v(g)\, \overline{W}_{F,v}(g)\, dg$$

Suppress v. At finite $v \neq v_1$, by Iwasawa decomposition and right K_v-invariance,

$$\int_{Z\backslash MN} \varphi(mn)\, \overline{W}_F(mn)\, dm\, dn$$

Further, with $Z\backslash MN \approx HN$ and $W_F(mn) = \psi(mnm^{-1})\, W_F(m)$ we get

$$\int_H \int_N \chi_0(m)\, \varphi(n)\, \overline{\psi}(mnm^{-1})\, \overline{W}_F(m)\, dm\, dn$$

Again, for m in the support of W_F and $n \in N\cap K$

$$\int_N \varphi(n)\, \overline{\psi}(mnm^{-1})\, dn \;=\; 1$$

So the integral becomes

$$\int_H \chi_0(m)\, \overline{W}_F(m)\, dm \;=\; \int_{k^\times} |y|^{s'}\, \overline{W}\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right) dy \;=\; (1-\overline{\alpha} q^{-s'})^{-1}\,(1-\overline{\beta} q^{-s'})^{-1} \;=\; L_v(s'+\frac{1}{2}, \overline{F})$$

For $v = v_1$, $\langle Pé, F\rangle$ unwinds to

$$\int_H \int_N \chi_0(m)\, \varphi(n)\, \overline{\psi}(mnm^{-1})\, \overline{W}_F(m)\, dm\, dn$$

Now $mnm^{-1} = \left(\begin{smallmatrix} 1 & xy\\ 0 & 1\end{smallmatrix}\right)$. So the integral becomes

$$\int_k \int_{k^\times} \overline{\psi}(xy)\, |y|^{s'}\, \overline{W}\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right) \varphi(x)\, dy\, dx$$

We will first consider the integral in y

$$\int_{k^\times} \overline{\psi}(xy)\,|y|^{s'}\,\overline{W}\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy$$

Denote the spherical Whittaker function by $W = \overline{W}_F = W_{\overline{F}}$. For $x \in \mathfrak{o}$ the integral is

$$\sum_{n=0}^{\infty} q^{-ns'}\,\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \;=\; (1-\alpha q^{-s'})^{-1}\,(1-\beta q^{-s'})^{-1} \;=\; L_v(s'+\frac{1}{2},\overline{F})$$

For $x \notin \mathfrak{o}$ we first evaluate $\int_{\mathfrak{o}^\times} \overline{\psi}(xy)\,dy$.

Let $ord(x) = m$.

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy)\,dy = \begin{cases} 1 & \text{(for } ord(y) \geq -ord(x)\text{)} \\ -\frac{1}{q-1} & \text{(for } ord(y) = -ord(x)-1\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

So, for $x \notin \mathfrak{o}$,

$$\int_{k^\times} \overline{\psi}(xy)\,|y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy$$

$$= \int_{ord(y)\geq -ord(x)} |y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy \;-\; \frac{1}{q-1}\int_{ord(y)=-ord(x)-1} |y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy$$

Now the whole integral is

$$\int_k \int_{k^\times} \overline{\psi}(xy)\,|y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) \varphi(x)\,dy\,dx$$

Again, the sub-integral over $x \in \mathfrak{o}$ evaluates to

$$(1-\alpha q^{-s'})^{-1}\,(1-\beta q^{-s'})^{-1} = L_v(s'+\frac{1}{2},f)$$

The sub-integral over $x \notin \mathfrak{o}$ becomes

$$\int_{ord(x)<0} |x|^{-w'} \int_{ord(y)\geq -ord(x)} |y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy\,dx$$
$$-\frac{1}{q-1}\int_{ord(x)<0} |x|^{-w'} \int_{ord(y)=-ord(x)-1} |y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy\,dx$$

First,

$$\int_{ord(x)<0} |x|^{-w'} \int_{ord(y)\geq -ord(x)} |y|^{s'}\,W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) dy\,dx$$

$$= \frac{q-1}{q}\int_{ord(x)<0} |x|^{1-w'} \int_{ord(y)\geq -ord(x)} |y|^{s'}\, W\big(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\big)\, dy\, dx$$

$$= \frac{q-1}{q}\sum_{m=1}^{\infty} q^{m(1-w')} \cdot \sum_{n\geq -m} q^{-ns'}\, \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

$$= \frac{q-1}{q}\sum_{m=1}^{\infty} q^{m(1-w')} \cdot (\alpha-\beta)^{-1}\big[\frac{\alpha^{1-m}q^{s'm}}{1-\alpha q^{-s'}} - \frac{\beta^{1-m}q^{s'm}}{1-\beta q^{-s'}}\big]$$

$$= \frac{q-1}{q} \cdot (\alpha-\beta)^{-1}\Big[(1-\alpha q^{-s'})^{-1}\, \alpha \sum_{m=1}^{\infty}(\alpha^{-1}q^{1-w'+s'})^m -$$

$$(1-\beta q^{-s'})^{-1}\, \beta \sum_{m=1}^{\infty}(\beta^{-1}q^{1-w'+s'})^m\Big]$$

$$= \frac{q-1}{q}\cdot(\alpha-\beta)^{-1}\big[\frac{q^{1-w'+s'}}{(1-\alpha q^{-s'})(1-\alpha^{-1}q^{1-w'+s'})} - \frac{q^{1-w'+s'}}{(1-\beta q^{-s'})(1-\beta^{-1}q^{1-w'+s'})}\big]$$

$$= \frac{(q-1)(q^{-w'} - \dfrac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha q^{-s'})\,(1-\beta q^{-s'})\,(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})}$$

$$= L_{v_1}(s'+\tfrac{1}{2}, f) \cdot \frac{(q-1)(q^{-w'} - \dfrac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})}$$

For $ord(y) = -ord(x) - 1$, write y as $y = \frac{t}{\varpi x}$ $(t \in \mathfrak{o}^{\times})$. Then

$$\frac{1}{q-1} \cdot \int_{ord(x)<0} |x|^{-w'} \cdot \int_{ord(y)=-ord(x)-1} |y|^{s'}\, W\big(\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\big)\, dy\, dx$$

$$= \frac{1}{q-1} \cdot \int_{ord(x)<0} |x|^{-w'} \cdot |\frac{1}{\varpi x}|^{s'}\, W\big(\begin{smallmatrix} \frac{1}{\varpi x} & 0 \\ 0 & 1\end{smallmatrix}\big)\, dx$$

$$= \frac{q-1}{q} \cdot \frac{1}{q-1} \cdot \int_{ord(x)<0} |x|^{1-w'} \cdot |\frac{1}{\varpi x}|^{s'}\, W\big(\begin{smallmatrix} \frac{1}{\varpi x} & 0 \\ 0 & 1\end{smallmatrix}\big)\, dx$$

$$= \frac{1}{q} \cdot \int_{ord(x)>0} |x|^{w'-1} \cdot |\frac{x}{\varpi}|^{s'}\, W\big(\begin{smallmatrix} \frac{x}{\varpi} & 0 \\ 0 & 1\end{smallmatrix}\big)\, dx$$

Replace x by ϖx and with $ord(x) = m$

$$\frac{1}{q} \cdot q^{1-w'} \int_{ord(x)\geq 0} |x|^{w'-1} \cdot |x|^{s'} \cdot W\big(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\big)\, dx$$

$$= q^{-w'} \cdot \sum_{m=0}^{\infty} q^{-m(w'-1+s')} \cdot \frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}$$

$$= q^{-w'} \cdot (1-\alpha q^{1-w'-s'})^{-1}\,(1-\beta q^{1-w'-s'})^{-1} \;=\; q^{-w'}\, L(s'+w'-\tfrac{1}{2}, F)$$

So, for $v = v_1$, the v^{th} local factor of $\langle P\acute{e}, F\rangle$ is

$$\frac{1}{(1-\alpha q^{-s'})\,(1-\beta q^{-s'})} + \frac{(q-1)(q^{-w'} - \dfrac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha q^{-s'})\,(1-\beta q^{-s'})\,(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})}$$
$$- \frac{1}{q^{w'}\,(1-\alpha q^{1-w'-s'})\,(1-\beta q^{1-w'-s'})}$$
$$= L_v(s'+\tfrac{1}{2},\overline{F}) + \frac{(q-1)(q^{-w'} - \dfrac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})} \cdot L_v(s'+\tfrac{1}{2},\overline{F}) \;-\; q^{-w'}L_v(s'+w'-\tfrac{1}{2},\overline{F})$$

For infinite v, by formulas (4.2) and (4.3) in [Diaconu-Garrett 2008] the v^{th} local factor of $\langle P\acute{e}, F\rangle$ is $\mathcal{G}(\frac{1}{2}+i\overline{\mu}_{F,v}; s', w)$, where up to a constant, for $v \approx \mathbb{R}$,

$$\mathcal{G}_v(s;s',w) = \pi^{-s'}\,\frac{\Gamma(\dfrac{s'+1-s}{2})\,\Gamma(\dfrac{s'+w-s}{2})\,\Gamma(\dfrac{s'+s}{2})\,\Gamma(\dfrac{s'+w+s-1}{2})}{\Gamma(\dfrac{w}{2})\,\Gamma(s'+\dfrac{w}{2})}$$

and at $v \approx \mathbb{C}$,

$$\mathcal{G}_v(s;s',w) = 2\pi^{-2s'}\,\frac{\Gamma(s'+1-s)\,\Gamma(s'+w-s)\,\Gamma(s'+s)\,\Gamma(s'+w+s-1)}{\Gamma(w)\,\Gamma(2s'+w)}$$

Group the archimedean factors as

$$\mathcal{G}_{F_\infty}(s',w) = \prod_{v|\infty}\mathcal{G}_v(\frac{1}{2}+i\overline{\mu}_{F,s'};s',w)$$

and let all ambiguous constants be absorbed into $\overline{\rho}_F$. Then, for cuspforms F, the cuspidal part of the spectral decomposition of the Poincaré series is

$$\textstyle\sum_F \langle P\acute{e},F\rangle\cdot F \;=\; \sum_F \overline{\rho}_F\,\mathcal{G}_{F_\infty}(s',w)\;\cdot\;[L_v(s'+\frac{1}{2},\overline{F}) + L_{v_1}(s'+\frac{1}{2},\overline{F}) +$$
$$\frac{(q-1)(q^{-w'} - \dfrac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})}\;\cdot\;L_{v_1}(s'+\tfrac{1}{2},\overline{F}) \;-\; q^{-w'}L_{v_1}(s'+w'-\tfrac{1}{2},\overline{F})]\;\cdot\;F$$

There is no residual spectrum since residual automorphic forms on $GL(2)$ are associated to one-dimensional representations which have no Whittaker models.

4.1.2 The continuous part

Subtract a suitable Eisenstein series from the Poincaré series and denote the resulting function by $P\acute{e}^*$. This function is L^2 and has sufficient decay to be integrated against an Eisenstein series (see section 4 in [Diaconu-Garrett 2008]). The leading term is:

$$\int_{N_{\mathbb{A}}} \varphi \;=\; \Big(\int_{N_\infty} \varphi_\infty \;\cdot\; \Big[\int_{N_{v\neq v_1}} \varphi_{v\neq v_1} \;\cdot\; \int_{N_{v_1}} \varphi_{v_1}\Big]\Big) \;\cdot\; E_{s'+1,1}$$

where, as in (4.16) in [Diaconu-Garrett 2008]

$$\int_{N_v} \varphi_v \;=\; \begin{cases} \sqrt{\pi}\frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} & (v \approx \mathbb{R}) \\ 2\pi(w-1)^{-1} & (v \approx \mathbb{C}) \end{cases}$$

For $v \neq v_1$,

$$\int_{N_v} \varphi_v \, dn \;=\; \int_{k_v} 1 \, dx \;=\; 1$$

and $\displaystyle\int_{N_{v_1}} \varphi_{v_1} \, dn \;=\; \int_{x\in\mathfrak{o}_v} 1 \, dx \;+\; \int_{x\notin\mathfrak{o}_v} |x|^{-w'} \, dx$

$$= 1 \;+\; \frac{q-1}{q} \;\cdot\; \sum_{m=1}^{\infty} (q^m)^{1-w'} \;=\; 1 \;+\; \frac{q-1}{q} \;\cdot\; \frac{q^{1-w'}}{1-q^{1-w'}} \;=\; \frac{1-q^{-w'}}{1-q^{1-w'}}$$

So the leading term is

$$\int_{N_\infty} \varphi_\infty \;\cdot\; \frac{1-q^{-w'}}{1-q^{1-w'}} \;\cdot\; E_{s'+1,1}$$

The continuous part of the spectral decomposition of $P\acute{e}$ is

$$\frac{1}{4\pi i\kappa} \sum_{\chi} \int_{Re(s)=\frac{1}{2}} \langle P\acute{e}^*, E_{s,\chi}\rangle \cdot E_{s,\chi} \, ds \quad (\text{where } \; \kappa = meas(\mathbb{J}^1/k^\times))$$

Thus, the spectral decomposition of the Poincaré series is

$$\begin{aligned} P\acute{e} &= \Big(\int_{N_\infty} \varphi_\infty\Big) \;\cdot\; \frac{1-q^{-w'}}{1-q^{1-w'}} \;\cdot\; E_{s'+1,1} \\ &+ \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(s',w) \;\cdot\; [L_v(s'+\frac{1}{2},\overline{F}) + L_{v_1}(s'+\frac{1}{2},\overline{F}) + \\ &\frac{(q-1)(q^{-w'} - \dfrac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})} \;\cdot\; L_{v_1}(s'+\tfrac{1}{2},\overline{F}) \;-\; q^{-w'} L_{v_1}(s'+w'-\tfrac{1}{2},\overline{F})] \;\cdot\; F \\ &+ \tfrac{1}{4\pi i\kappa} \sum_{\chi} \int_{Re(s)=\frac{1}{2}} \langle P\acute{e}^*, E_{s,\chi}\rangle \cdot E_{s,\chi} \, ds \end{aligned}$$

As in section 4 in [Diaconu-Garrett 2008],

$$\langle P\acute{e}^*, E_{s,\chi}\rangle = \Big(\int_{Z_\infty\backslash G_\infty} \varphi_\infty \;\cdot\; \overline{W}^E_{s,\chi,\infty}\Big) \;\cdot\; \Big(\prod_{v<\infty} \int_{Z_v\backslash G_v} \varphi_v(g_v) \;\cdot\; \overline{W}^E_{s,\chi,v}(g_v)\Big) \, dg_v$$

where

$$\int_{Z_v\backslash G_v} \varphi_\infty \;\cdot\; \overline{W}^E_{s,\chi,v} = \begin{cases} \dfrac{\mathcal{G}_v(s,s',w)}{\pi^{-s}\Gamma(s)} & (v \approx \mathbb{R}) \\ \dfrac{\mathcal{G}_v(s,s',w)}{2\pi^{-2s-1}\Gamma(2s)} & (v \approx \mathbb{C}) \end{cases}$$

and for finite $v \neq v_1$,

$$\int_{Z_v\backslash G_v} \varphi_v(g_v) \cdot \overline{W}^E_{s,\chi,v}(g_v)$$
$$= |\mathfrak{d}_v|_v^{\frac{1}{2}} \cdot \frac{L_v(s'+\overline{s},\overline{\chi}_v) \cdot L_v(s'+1-\overline{s},\chi_v)}{L_v(2\overline{s},\overline{\chi}_v^2)} \cdot |\mathfrak{d}_v|_v^{-(s'+1-\overline{s})} \cdot \overline{\chi}_v(\mathfrak{d}_v)$$

where $\mathfrak{d}$ is the idele with v^{th} component $\mathfrak{d}_v$ at finite place v and component 1 at archimedean places. For finite $v = v_1$,

$$\int_{Z_v\backslash G_v} \varphi_v(g_v) \cdot \overline{W}^E_{s,\chi,v}(g_v) = \int_k \int_{k^\times} |y|^{s'}\, \overline{\psi}(xy)\, \overline{W}^E_{s,\chi}\big(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\big) \cdot \varphi(x)\, dy\, dx$$

Define an Eisenstein series by

$$E(g) = \sum_{\gamma \in P_k\backslash G_k} \eta(\gamma g)$$

for η left P_k-invariant, left M_k-invariant and left $N_{\mathbb{A}}$-invariant. Present the vectors η_v in a different form, namely

$$\eta_v(pk) = \Big|\frac{a}{d}\Big|_v^s \cdot \chi_v\big(\frac{a}{d}\big) \quad (\text{for } p = \big(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\big) \in P_v,\, k \in K_v)$$

Let ϕ_v be any Schwartz function on k_v^2, invariant under k_v and put

$$\eta'_v(g) = \chi_v(\det g)|\det g|_v^s \cdot \int_{k_v^\times} \chi_v^2(t)|t|_v^{2s} \cdot \phi_v(t \cdot e_2 \cdot g)\, dt$$

where $e_2 = e_{2,v}$ is the second basis element in k_v^2. η'_v has the same left P_v-equivariance as η_v:

$$\eta'_v\big(\big(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\big) \cdot g\big) = \Big|\frac{a}{d}\Big|_v^s \cdot \chi_v\big(\frac{a}{d}\big) \cdot \eta'_v(g)$$

For ϕ_v invariant under K_v, the function η'_v is right K_v-invariant. So as in Appendix 2 in [Diaconu-Garrett 2008], $\eta'_v(g) = \eta'_v(1) \cdot \eta_v(g)$ since $\eta_v(1) = 1$), and

$$\eta'_v(1) = \int_{k_v^2} \chi_v^2(t)\, |t|_v^{2s} \cdot \phi_v(t \cdot e_2 \cdot 1)\, dt = \zeta_v(2s, \chi^2, \phi(0,*))$$

Thus, it suffices to compute the local Mellin transform of

$$\eta'_v(1) \cdot W^E_{s,\chi,v}(m) = \int_{N_v} \overline{\psi}(n) \cdot \eta'_v(w_0 n m)\, dn$$
$$= \chi(y)|y|^s \cdot \int_{N_v} \overline{\psi}(n) \int_{k_v^\times} \chi_v^2(t)\, |t|_v^{2s} \cdot \phi_v(t \cdot e_2 \cdot w_0 \cdot nm)\, dt\, dn$$
$$= \chi(y)|y|^s \cdot \int_{k_v} \overline{\psi}(x') \int_{k_v^\times} \chi_v^2(t)\, |t|_v^{2s} \cdot \phi_v(tx', ty)\, dt\, dx' \quad (\text{with } m = \big(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\big))$$

At finite primes, take $\phi(t,x') = ch_{\mathfrak{o}_v}(t) \cdot ch_{\mathfrak{o}_v}(x')$ where ch_X is the characteristic function of a set X.

Then $\eta'_v(1) = \zeta_v(2s,\chi^2, ch_{\mathfrak{o}_v}) = L_v(2s,\chi^2)$ and:

$$\eta'_v(1)\cdot W^E_{s,\chi,v}\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right) \;=\; \chi(y)|y|^s \cdot \int_{k_v} \overline{\psi}(x')\, ch_{\mathfrak{o}_v}(tx')\cdot \int_{k_v^\times} \chi_v^2(t)\,|t|_v^{2s}\, ch_{\mathfrak{o}_v}(ty)\,dt\,dx'$$
$$= \chi(y)|y|^s\, meas(\mathfrak{o}_v) \int_{k_v^\times} ch_{\mathfrak{o}_v^*}(\frac{1}{t})\,\chi_v^2(t)\,|t|_v^{2s-1}\, ch_{\mathfrak{o}_v}(ty)\,dt$$
$$= |\mathfrak{d}_v|^{\frac{1}{2}} \cdot \chi(y)|y|^s \int_{k_v^\times} ch_{\mathfrak{o}_v^*}(\frac{1}{t})\,\chi_v^2(t)\,|t|_v^{2s-1}\, ch_{\mathfrak{o}_v}(ty)\,dt$$

where $\mathfrak{d}_v \in k_v^\times$ is such that $(\mathfrak{o}_v^*)^{-1} = \mathfrak{d}_v \cdot \mathfrak{o}_v$. So, omitting $|\mathfrak{d}_v|^{\frac{1}{2}}$ for now,

$$\int_k \int_{k^\times} |y|^{s'}\,\overline{\psi}(xy)\,\overline{W}^E_{s,\chi}\left(\begin{smallmatrix} y & 0\\ 0 & 1\end{smallmatrix}\right) \cdot \varphi(x)\,dy\,dx$$
$$= \int_k \int_{k^\times} |y|^{s'}\,\overline{\psi}(xy)\cdot \big(\chi(y)|y|^s \int_{k_v^\times} ch_{\mathfrak{o}_v^*}(\frac{1}{t})\,\chi_v^2(t)\,|t|_v^{2s-1}\, ch_{\mathfrak{o}_v}(ty)\,dt\big)\cdot\varphi(x)\,dy\,dx$$

Consider the integrals in y and t. Replace y by $\frac{y}{t}$ and t by $\frac{1}{t}$ to get

$$\int_{k^\times}\int_{k^\times} \overline{\psi}(xyt)\cdot\chi(y)\,|y|^{s+s'}\,ch_{\mathfrak{o}_v}(y)\cdot ch_{\partial_v\mathfrak{o}_v}(t)\cdot\overline{\chi}(t)\,|t|^{s'+1-s}\,dt\,dy$$

First consider the integral in y:

$$\int_{k^\times} \overline{\psi}(xty)\cdot\chi(y)\,|y|^{s+s'}\,ch_{\mathfrak{o}_v}(y)\,dy$$

For $x \in \mathfrak{o}^\times$, ψ is trivial on $\mathfrak{o}$, so we get

$$\int_{\mathfrak{o}^\times}\chi(y)\,|y|^{s+s'}\,dy\cdot\int_{k^\times} ch_{\partial_v\mathfrak{o}_v}(t)\cdot\overline{\chi}(t)\,|t|^{s'+1-s}\,dt$$
$$= L_{v_1}(s+s',\chi)\cdot L_{v_1}(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+1-s)}\,\chi(\mathfrak{d}_v)$$

For $x \notin \mathfrak{o}^\times$,

$$\int_{\mathfrak{o}^\times}\overline{\psi}(xty)\,dy = \begin{cases} 1 & (ord(y) \ge -ord(x) - ord(t)) \\ -\frac{1}{q-1} & (ord(y) = -ord(x) - ord(t) - 1) \\ 0 & (\text{otherwise}) \end{cases}$$

So

$$\int_{k^\times}\overline{\psi}(xty)\cdot\chi(y)\,|y|^{s+s'}\,ch_{\mathfrak{o}_v}(y)\,dy \;=\; \int_{\mathfrak{o}^\times}\overline{\psi}(xty)\cdot\chi(y)\,|y|^{s+s'}\,dy$$
$$= \int_{ord(y)\ge -ord(x)-ord(t)} \chi(y)\,|y|^{s+s'}\,dy - \frac{1}{q-1}\int_{ord(y) = -ord(x)-ord(t)-1} \chi(y)\,|y|^{s+s'}\,dy$$

The entire integral in t and y is:

$$\int_{x\notin\mathfrak{o}} \varphi(x) \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt \cdot$$
$$\Big[\int_{ord(y)\geq -ord(x)-ord(t)} \chi(y)\,|y|^{s+s'}\,dy - \frac{1}{q-1}\int_{ord(y)=-ord(x)-ord(t)-1} \chi(y)\,|y|^{s+s'}\,dy\Big]$$

First take

$$\int_{x\notin\mathfrak{o}} \varphi(x) \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt \cdot \int_{ord(y)\geq -ord(x)-ord(t)} \chi(y)\,|y|^{s+s'}\,dy\,dx$$
$$= \frac{q-1}{q}\sum_{m=1}^{\infty}(q^m)^{1-w'} \cdot \sum_{n\geq -m-r}^{\infty}(q^{-n})^{s+s'} \cdot \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt$$

(where $ord(t)=r$ and $\chi(y)$ is omitted for now)

$$= \frac{q-1}{q}\sum_{m=1}^{\infty}\frac{(q^{1-w'})^m \cdot (q^{s+s'})^m \cdot (q^r)^{s+s'}}{1-q^{-(s+s')}} \cdot \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt$$
$$= \frac{(q-1)\,q^{1-w'+s+s'}}{q(1-q^{1-w'+s+s'})(1-q^{-s-s'})} \cdot \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,|t|^{-s-s'}\,dt$$
$$= \frac{(q-1)\,q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})} \cdot L_{v_1}(1-2s,\overline{\chi})$$

Next we take

$$-\frac{1}{q-1}\int_{x\notin\mathfrak{o}} \varphi(x) \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt \cdot \int_{ord(y)=-ord(x)-ord(t)-1} \chi(y)\,|y|^{s+s'}\,dy\,dx$$

Since $ord(y) = -ord(x) - ord(t) - 1$, y can be written as $y = \frac{1}{\varpi t x}$. So the entire integral becomes an integral in t and x as follows:

$$-\frac{1}{q-1}\int_{x\notin\mathfrak{o}} \varphi(x) \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt \cdot \int_{ord(y)=-ord(x)-ord(t)-1} \chi(y)\,|y|^{s+s'}\,dy\,dx$$
$$= -\frac{1}{q-1} \cdot \frac{q-1}{q}\int_{x\notin\mathfrak{o}} |x|^{1-w'} \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{s'+1-s}\,dt \cdot \Big|\frac{1}{\varpi x t}\Big|^{s+s'}\,dx$$
$$= -q^{s+s'-1}\int_{|x|>1} |x|^{1-w'-s-s'} \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)\,|t|^{1-2s}\,dt\,dx$$
$$= -q^{s+s'-1}\sum_{m=1}^{\infty}(q^m)^{1-w'-s-s'} \cdot L_{v_1}(1-2s,\overline{\chi})$$
$$= -\frac{q^{s+s'-1} \cdot q^{1-w'-s-s'}}{1-q^{1-w'-s-s'}} \cdot L_{v_1}(1-2s,\overline{\chi}) = -\frac{q^{-w'}}{1-q^{1-w'-s-s'}} \cdot L_{v_1}(1-2s,\overline{\chi})$$

Thus adding up we get

$$L_{v_1}(1-2s,\overline{\chi}) \cdot \Big[\frac{(q-1)\,q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})} - \frac{q^{-w'}}{1-q^{1-w'-s-s'}}\Big]$$

Thus at finite primes $v = v_1$, the integral is

$$L_{v_1}(s+s',\chi)\cdot L_{v_1}(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_{v_1}|^{-(s'+1-s)}\,\chi(\mathfrak{d}_{v_1})+$$

$$L_{v_1}(1-2s,\overline{\chi})\cdot\Big[\frac{(q-1)\,q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})}-\frac{q^{-w'}}{1-q^{1-w'-s-s'}}\Big]\cdot|\mathfrak{d}_{v_1}|^{-(1-2s)}\,\chi(\mathfrak{d}_{v_1})$$

Dividing by η'_v and putting back the measure constant $|\mathfrak{d}_v|^{\frac{1}{2}}$, for $v = v_1$,

$$|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\int_{k^\times}\int_k|y|^{s'}\,\overline{\psi}(xy)\,\overline{W}^E_{s,\chi}\big(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\big)\cdot\varphi(x)\,dy\,dx$$
$$=\frac{L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+1-s)}\cdot|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\chi(\mathfrak{d}_v)}{L_v(2s,\chi^2)}$$
$$+\frac{L_v(1-2s,\overline{\chi})\cdot[\frac{(q-1)\,q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})}-\frac{q^{-w'}}{1-q^{1-w'-s-s'}}]\cdot|\mathfrak{d}_v|^{-(1-2s)}\cdot|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\chi(\mathfrak{d}_v)}{L_v(2s,\chi^2)}$$

Replacing s by $1-s$ and χ by $\overline{\chi}$

$$|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\int_{k^\times}\int_k|y|^{s'}\,\overline{\psi}(xy)\,\overline{W}^E_{1-s,\overline{\chi}}\big(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\big)\cdot\varphi(x)\,dy\,dx$$
$$=\frac{L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})}\cdot\chi(\mathfrak{d}_v)}{L_v(2-2s,\overline{\chi}^2)}$$
$$+\frac{L_v(2s-1,\overline{\chi})\cdot[\frac{(q-1)\,q^{1-w'+s+s'}}{(1-q^{2-w'-s+s'})(1-q^{-1+s-s'})}-\frac{q^{-w'}}{1-q^{-w'+s-s'}}]\cdot|\mathfrak{d}_v|^{\frac{3}{2}-2s}\cdot\chi(\mathfrak{d}_v)}{L_v(2-2s,\chi^2)}$$

So the spectral decomposition of the Poincaré series is:

$$P\acute{e}=\Big(\int_{N_\infty}\varphi_\infty\Big)\cdot\frac{1-q^{-w'}}{1-q^{1-w'}}\cdot E_{s'+1,1}$$
$$+\sum_F\overline{\rho}_F\mathcal{G}_{F_\infty}(s',w)\cdot[L_v(s'+\frac{1}{2},\overline{F})+L_{v_1}(s'+\frac{1}{2},\overline{F})+$$
$$\frac{(q-1)(q^{-w'}-\frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})\,(1-\beta^{-1}q^{1-w'+s'})}\cdot L_{v_1}(s'+\tfrac{1}{2},\overline{F})-$$
$$q^{-w'}L_{v_1}(s'+w'-\tfrac{1}{2},\overline{F})]\cdot F+\tfrac{1}{4\pi i\kappa}\sum_\chi\int_{\Re(s)=\frac{1}{2}}\Big(\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot\overline{W}^E_{s,\chi,\infty}\Big)\cdot$$
$$\Big(\frac{L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})}\cdot\overline{\chi}(\mathfrak{d}_v)}{L_v(2-2s,\overline{\chi}^2)}+$$

$$\frac{L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})}\cdot\overline{\chi}(\mathfrak{d}_v)}{L_v(2-2s,\overline{\chi}^2)}+$$

$$\frac{L_v(2s-1,\overline{\chi})\cdot[\frac{(q-1)\,q^{1-w'-s+s'}}{(1-q^{2-w'-s+s'})(1-q^{-1+s-s'})}-\frac{q^{-w'}}{1-q^{-w'+s-s'}}]\cdot|\mathfrak{d}_v|^{\frac{3}{2}-2s}\cdot\chi(\mathfrak{d}_v)}{L_v(2-2s,\chi^2)}\Big)\cdot E_{s,\chi}\,ds$$

where

$$\int_{N_v}\varphi_v=\begin{cases}\sqrt{\pi}\dfrac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} & (v\approx\mathbb{R})\\ 2\pi(w-1)^{-1} & (v\approx\mathbb{C})\end{cases}$$

and

$$\int_{Z_v\backslash G_v}\varphi_v\cdot\overline{W}^E_{s,\chi,v}=\begin{cases}\dfrac{\mathcal{G}_v(s,s',w)}{\pi^{-s}\Gamma(s)} & (v\approx\mathbb{R})\\ \dfrac{\mathcal{G}_v(s,s',w)}{2\pi^{-2s-1}\Gamma(2s)} & (v\approx\mathbb{C})\end{cases}$$

5.1 PRELIMINARIES TO SUBCONVEXITY

Fix a non-archimedean place v_1, and take $1<\beta'<2$. Recall that

$$\mathcal{K}_{v_1}(w',\chi_{v_1})\ll(q^N)^{-w'}$$

where $\mathcal{K}_{v_1}(w',\chi_{v_1})$ is the non-decoupled integral for finite prime v_1 at which χ has ramification with conductor q^N. Define

$$Z(w')=\sum_{\chi\in\hat{C}_{0,S}}\int_{-\infty}^{\infty}|L(\frac{1}{2}+it,f\otimes\chi)|^2\cdot(q^N)^{-w'}\cdot\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)\,dt$$

This is a modified function obtained from

$$I(0,w')=\sum_{\chi}\frac{1}{2\pi i}\int_{-\infty}^{\infty}|L(\frac{1}{2}+it,f\otimes\chi)|^2\cdot\mathcal{K}_{v_1}(w',\chi_{v_1})\cdot\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)\,dt$$

by taking the asymptotic formula for $\mathcal{K}_{v_1}(w',\chi_{v_1})$. $Z(w')$ is absolutely convergent for $\Re(w')>1$ (see Section 5 in [Diaconu-Garrett 2008]). We will prove the meromorphic continuation and polynomial growth of $Z(w')$, yielding subconvexity bounds in the χ-depth-aspect.

5.1.1 Meromorphic continuation of $Z(w')$

Theorem 5.1 The function

$$Z(w')=\sum_{\chi\in\hat{C}_{0,S}}\int_{-\infty}^{\infty}|L(\frac{1}{2}+it,f\otimes\chi)|^2\cdot(q^N)^{-w'}\cdot\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)\,dt$$

where the sum is over characters ramified at the finite prime v_1 with conductor q^N, and $1 < \beta' \leq 2$, $\Re(w') > 1$, has analytic continuation to $\Re(w') > \frac{11}{18}$, except for $w' = 1$ where it has a pole of order 1.

Proof. Let $w' = \delta + i\eta$. Split $Z = Z_1 + Z_2$ where, for a positive constant C,

$$Z_1(w') = \sum_{\chi \in \hat{C}_{0,S} : q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt$$

We first show that $Z_1(w')$ has analytic continuation by showing that it is holomorphic for $\delta > 0$. Now

$$|Z_1(w')| \leq \sum_{\chi : q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f \otimes \chi)|^2 \cdot |(q^N)^{-w'}| \cdot |\mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)|\, dt$$

$\mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)$ is positive (see Section 4 in [Diaconu-Garrett 2009]). So

$$|Z_1(w')| \leq \sum_{\chi : q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f \otimes \chi)|^2 \cdot (q^N)^{-\delta} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt$$

Since

$$(q^N)^{-\delta} \ll_{\beta', C} (q^N)^{-\beta'}$$

then

$$|Z_1(w')| \ll \sum_{\chi : q^N \ll C} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f \otimes \chi)|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt$$
$$< \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f \otimes \chi)|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt = Z(\beta')$$

which is convergent for $\Re(w') < \frac{2}{9}$. Thus, $Z_1(w')$ is holomorphic for $\Re(w') = \delta > 0$ (in particular for $\Re(w') > \frac{11}{18}$).

Now we prove that $Z_2(w')$ has analytic continuation. Consider

$$I(s', w', \beta') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s, f \otimes \chi) \cdot L(s'+1-s, \overline{f} \otimes \overline{\chi}) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, s', \beta', \chi)\, dt$$

where

$$\mathcal{K}_\infty(s, s', \beta', \chi) = \prod_{v | \infty} \mathcal{K}_v(s, s', \beta', \chi_v)$$

Recall that this expression is obtained from the integral representation

$$\int_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} P\acute{e}\, |f|^2\, dg$$

where $P\acute{e}(g)$ converges absolutely and locally uniformly for $\Re(s') > 1$ and $\Re(w') > 1$. Also recall from the spectral decomposition that $P\acute{e}$ has meromorphic continuation to $\Re(w') > \frac{11}{18}$ with a simple pole at $w' = 1$. Thus

$$I(0,w',\beta') = \sum_{\chi\in\hat{C}_{0,S}} \frac{1}{2\pi i}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot \mathcal{K}_{v_1}(w',\chi_{v_1}) \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt$$

is holomorphic for $\Re(w') > \frac{11}{18}$ except at $w' = 1$ where there is a simple pole.

In the region of absolute convergence for $\Re(w') = \delta > 1$, write $I = I_1 + I_2$ where

$$I_1(0,w',\beta') = \sum_{\chi:q^N\ll C} \tfrac{1}{2\pi i}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot \mathcal{K}_{v_1}(w',\chi_{v_1}) \cdot \mathcal{K}_\infty(\tfrac{1}{2}+it, 0, \beta', \chi)\, dt$$

Now

$$I(0,w',\beta') \;=\; I_1(0,w',\beta') + \sum_{\chi:q^N\gg C} \tfrac{1}{2\pi i}\textstyle\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot C' \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt$$

where the constant is

$$C' = \frac{q}{q-1} \;\cdot\; \frac{1-|\alpha|^2|\beta|^2 q^{-2w'}}{(1-|\alpha|^2q^{-w'})(1-|\beta|^2q^{-w'})(1-\overline{\alpha}\beta q^{-w'})(1-\alpha\overline{\beta}q^{-w'})}$$

since

$$\mathcal{K}_{v_1}(w',\chi_{v_1}) = \frac{q^{1-Nw'}}{q-1} \cdot \frac{1-|\alpha|^2|\beta|^2 q^{-2w'}}{(1-|\alpha|^2q^{-w'})(1-|\beta|^2q^{-w'})(1-\overline{\alpha}\beta q^{-w'})(1-\alpha\overline{\beta}q^{-w'})}$$

Therefore

$$I(0,w',\beta') = I_1(0,w',\beta') + C' \cdot Z_2(w')$$

Thus, to show that $Z_2(w')$ has analytic continuation, it suffices to show that $I_1(0,w',\beta')$ is absolutely convergent for $\Re(w') > \frac{11}{18}$.

$$\begin{aligned}
&I_1(0,w',\beta') = \\
&\sum_{\chi:q^N\ll C} \frac{1}{2\pi i}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \,\cdot\, \mathcal{K}_{v_1}(w',\chi_{v_1}) \,\cdot\, \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt \\
&\ll \sum_{\chi:q^N\ll C} \frac{1}{2\pi i}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \,\cdot\, |(q^N)^{-w'}| \,\cdot\, \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt \\
&\ll \sum_{\chi:q^N\ll C} \frac{1}{2\pi i}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \,\cdot\, (q^N)^{-\beta} \,\cdot\, \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt \\
&< \sum_{\chi\in\hat{C}_{0,S}} \frac{1}{2\pi i}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \,\cdot\, (q^N)^{-\beta} \,\cdot\, \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt
\end{aligned}$$

$$= Z(\beta')$$

which is convergent for $\Re(w') > \frac{2}{9}$. Thus $Z_2(w')$ is absolutely convergent for $\Re(w') > \frac{11}{18}$, proving the theorem.

□

5.1.2 Polynomial growth of $Z(w')$

Theorem 5.2 For every fixed small positive ϵ, the generating function

$$Z(w') = \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi)\, dt$$

has polynomial growth in the conductor q^N for $\frac{11}{18} + \epsilon \le \Re(w') \le 1 + \epsilon$; that is, on the vertical line $\Re(w') = \frac{11}{18} + \epsilon$, for computable $\gamma > 0$ independent of β',

$$Z(w') \ll_{\epsilon,\beta'} (q^N)^\gamma$$

For $\Re(s'), \Re(w') > 1$,

$$I(s', w', \beta') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s, f \otimes \chi) \cdot L(s' + 1 - s, \overline{f} \otimes \overline{\chi}) \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(s, s', \beta', \chi)\, dt$$

where

$$\mathcal{K}_\infty(s, s', \beta', \chi) = \prod_{v|\infty} \mathcal{K}_v(s, s', \beta', \chi_v)$$

In the region of absolute convergence $I = I_1 + I_2$ where

$$I_1(0, w', \beta') = \sum_{\chi : q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \mathcal{K}_{v_1}(w', \chi_{v_1}) \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi)\, dt$$

and

$$I_2(0, w', \beta') = C' \cdot Z_2(w') = C'[Z(w') - Z_1(w')]$$

Recall that $Z_1(w')$ is holomorphic in the half-plane $\Re(w') > \frac{11}{18}$, and that C' is the positive constant where $I(0, w', \beta')$ was cutoff. So $Z_1(w')$ has polynomial growth in q^N. Thus, the polynomial bound of $Z(w')$ will be deduced from that of $I_2(0, w', \beta')$.

In the spectral decomposition of $P\acute{e}$ set $s' = 0$ and obtain

$$P\acute{e} =$$

$$\lim_{s' \to 0} \Big(\int_{N_\infty} \varphi_\infty \Big) \cdot \frac{1 - q^{-w'}}{1 - q^{1-w'}} \cdot E_{s'+1,1} + \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(\beta') \cdot \Big[L_v(\frac{1}{2}, \overline{F}) +$$

$$L_{v_1}(\tfrac{1}{2}, \overline{F}) + \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'}}{\alpha\beta})}{(1 - \alpha^{-1} q^{1-w'})\,(1 - \beta^{-1} q^{1-w'})} \cdot L_{v_1}(\tfrac{1}{2}, \overline{F}) - q^{-w'} L_{v_1}(\tfrac{2w'-1}{2}, \overline{F}) \Big] \cdot F +$$

$$\sum_{\chi} \frac{\overline{\chi}(\mathfrak{d}_v)}{4\pi i\kappa} \int_{\Re(s)=\frac{1}{2}} \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \, \cdot \, \overline{W}^E_{s,\overline{\chi},\infty} \Big) \cdot$$

$$\Big(\frac{L_v(s,\chi) \, \cdot \, L_v(1-s,\overline{\chi}) \, \cdot \, |\mathfrak{d}_v|^{-(s-\frac{1}{2})}}{L_v(2-2s,\overline{\chi}^2)} + \frac{L_{v_1}(s,\chi) \, \cdot \, L_{v_1}(1-s,\overline{\chi}) \, \cdot \, |\mathfrak{d}_{v_1}|^{-(s-\frac{1}{2})}}{L_{v_1}(2-2s,\overline{\chi}^2)} +$$

$$\frac{L_{v_1}(2s-1,\overline{\chi}) \, \cdot \, \Big[\frac{(q-1)\, q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})} \, - \, \frac{q^{-w'}}{1-q^{-w'+s}} \Big] \, \cdot \, |\mathfrak{d}_{v_1}|^{\frac{3}{2}-2s}}{L_{v_1}(2-2s,\chi^2)} \Big) \, \cdot \, E_{s,\chi} \, ds$$

So

$$I(w') = I_{sing}(w') + I_{cusp}(w') + I_{cont}(w')$$

where

$$I_{sing}(w') = \lim\nolimits_{s'\to 0} \Big(\int_{N_\infty} \varphi_\infty \Big) \, \cdot \, \frac{1-q^{-w'}}{1-q^{1-w'}} \, \cdot \, \langle E_{s'+1,1}, |f|^2 \rangle$$

$$I_{cusp}(w') = \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(\beta') \, \cdot \, \Big[2L_v(\frac{1}{2},\overline{F}) + \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'})\,(1-\beta^{-1}q^{1-w'})} \, \cdot$$

$$L_v(\frac{1}{2},\overline{F}) \, - \, q^{-w'} L_v(\frac{2w'-1}{2},\overline{F}) \Big] \, \cdot \, \langle F, |f|^2 \rangle$$

$$I_{cont}(w') = \sum_{\chi} \frac{\overline{\chi}(\mathfrak{d}_v)}{4\pi i\kappa} \int_{\Re(s)=\frac{1}{2}} \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{1-s,\overline{\chi},\infty} \Big) \cdot \Big(\frac{2L_v(s,\chi) \, \cdot \, L_v(1-s,\overline{\chi}) \, \cdot \, |\mathfrak{d}_v|^{-(s-\frac{1}{2})}}{L_v(2-2s,\overline{\chi}^2)} +$$

$$\frac{L_v(2s-1,\overline{\chi}) \, \cdot \, \Big[\frac{(q-1)\, q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})} \, - \, \frac{q^{-w'}}{1-q^{-w'+s}} \Big] \, \cdot \, |\mathfrak{d}_v|^{\frac{3}{2}-2s}}{L_v(2-2s,\chi^2)} \Big) \cdot \langle E_{s,\chi}, |f|^2 \rangle \, ds$$

Let

$$\mathcal{M}_1(w') = \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'})\,(1-\beta^{-1}q^{1-w'})} \, \cdot \, L(\frac{1}{2},\overline{F}) \, - \, q^{-w'} L(\frac{2w'-1}{2},\overline{F})$$

and

$$\mathcal{M}_2(w') = \frac{(q-1)\, q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})} \, - \, \frac{q^{-w'}}{1-q^{-w'+s}}$$

Then define the auxiliary function $I^{aux}(w')$ by

$$I^{aux}(w') = \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(\beta') \, \cdot \, [2L(\frac{1}{2},\overline{F}) + \mathcal{M}_1^{aux}(w')] \, \cdot \, \langle F, |f|^2 \rangle +$$

$$\sum_{\chi} \frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa} \int_{\Re(s)=\frac{1}{2}} \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{1-s,\overline{\chi},\infty} \Big) \cdot \frac{2L(s,\chi) \, \cdot \, L(1-s,\overline{\chi}) \, \cdot \, |\mathfrak{d}|^{-(s-\frac{1}{2})}}{L(2-2s,\overline{\chi}^2)} +$$

$$\frac{L(2s-1,\overline{\chi}) \, \cdot \, \mathcal{M}_2^{aux}(w') \, \cdot \, |\mathfrak{d}|^{\frac{3}{2}-2s}}{L(2-2s,\chi^2)} \, \cdot \, \langle E_{s,\chi}, |f|^2 \rangle \, ds$$

where $\mathcal{M}_1^{aux}(w')$ and $\mathcal{M}_2^{aux}(w')$ are defined by

$$\mathcal{M}_1^{aux}(w') = \mathcal{M}_1(w') \cdot (q^N)^\gamma, \quad \mathcal{M}_2^{aux}(w') = \mathcal{M}_2(w') \cdot (q^N)^\gamma$$

where $\gamma > 0$, independent of β'. Define

$$H(w') = I(w') - I^{aux}(w') = \lim_{s' \to 0}\Big(\int_{N_\infty} \varphi_\infty\Big) \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot \langle E_{s'+1,1}, |f|^2\rangle + \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(\beta') \cdot [\mathcal{M}_1(w') - \mathcal{M}_1^{aux}(w')] \cdot \langle F, |f|^2\rangle + \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i \kappa} \int_{\Re(s)=\frac{1}{2}} \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{1-s,\overline{\chi},\infty}\Big) \cdot \frac{L(2s-1,\overline{\chi}) \cdot [\mathcal{M}_2(w') - \mathcal{M}_2^{aux}(w')] \cdot |\mathfrak{d}|^{\frac{3}{2}-2s}}{L(2-2s,\chi^2)} \cdot \langle E_{s,\chi}, |f|^2\rangle \, ds$$

Proposition 5.3 For ϵ sufficiently small, $H(w') = I(w') - I^{aux}(w')$ restricted to $\frac{11}{18} < \Re(w') \leq 1+\epsilon$, extends holomorphically to the whole vertical strip $-\epsilon \leq \Re(w') \leq 1+\epsilon$.

Proof. The first term in $H(w')$, i.e.

$$\lim_{s' \to 0}\Big(\int_{N_\infty} \varphi_\infty\Big) \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot \langle E_{s'+1,1}, |f|^2\rangle$$

is holomorphic in the strip $-\epsilon \leq \Re(w') \leq 1+\epsilon$, except at $w' = 0, 1$ where there are poles.
$\mathcal{M}_1(w') - \mathcal{M}_1^{aux}(w') = \mathcal{M}_1(w') - \mathcal{M}_1(w') \cdot (q^N)^\gamma = \mathcal{M}_1(w')[1-(q^N)^\gamma]$ and
$\mathcal{M}_2(w') - \mathcal{M}_2^{aux}(w') = \mathcal{M}_2(w') - \mathcal{M}_2(w') \cdot (q^N)^\gamma = \mathcal{M}_2(w')[1-(q^N)^\gamma]$
Since both $\mathcal{M}_1(w')$ and $\mathcal{M}_2(w')$ are holomorphic in the strip, then $H(w')$ is also holomorphic in the strip.

□

Proposition 5.4 For small $\epsilon > 0$, for $\frac{11}{18} + \epsilon \leq \Re(w') \leq 1+\epsilon$, or $\Re(w') = -\epsilon$,

$$I^{aux}(w') \ll_{\epsilon,\beta'} (q^N)^\gamma$$

Proof. Again,

$$I^{aux}(w') = \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(\beta') \cdot [2L(\frac{1}{2}, \overline{F}) + \mathcal{M}_1^{aux}(w') \cdot \langle F, |f|^2\rangle + \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i \kappa} \int_{\Re(s)=\frac{1}{2}} \Big(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{1-s,\overline{\chi},\infty}\Big) \cdot \frac{2L(s,\chi) \cdot L(1-s,\overline{\chi}) \cdot |\mathfrak{d}|^{-(s-\frac{1}{2})}}{L(2-2s,\overline{\chi}^2)} + \frac{L(2s-1,\overline{\chi}) \cdot \mathcal{M}_2^{aux}(w') \cdot |\mathfrak{d}|^{\frac{3}{2}-2s}}{L(2-2s,\chi^2)} \cdot \langle E_{s,\chi}, |f|^2\rangle \, ds$$

where

$$\mathcal{M}_1^{aux}(w') = \mathcal{M}_1(w') \cdot (q^N)^\gamma \quad \mathcal{M}_2^{aux}(w') = \mathcal{M}_2(w') \cdot (q^N)^\gamma$$

$$\mathcal{M}_1(w') = \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'})\,(1-\beta^{-1}q^{1-w'})} \cdot L(\tfrac{1}{2},\overline{F}) \; - \; q^{-w'} L(\tfrac{2w'-1}{2},\overline{F})$$

$$\mathcal{M}_2(w') = \frac{(q-1)\,q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})} - \frac{q^{-w'}}{1-q^{-w'+s}}$$

$$\mathcal{M}_1^{aux}(w') \ll (q^N)^\gamma \quad \mathcal{M}_2^{aux}(w') \ll (q^N)^\gamma$$

All other terms in $I^{aux}(w')$ are independent of the conductor q^N, and have a polynomial bound. Thus $I^{aux}(w') \ll_{\epsilon,\beta'} (q^N)^\gamma$.

□

Recall we are trying to prove a polynomial bound for $I_2(w')$ in the conductor q^N. Now

$$I_2(w') = I_2(w') - I^{aux}(w') + I^{aux}(w')$$

We have a polynomial bound for $I^{aux}(w')$, and now prove a polynomial bound for $I_2(w') - I^{aux}(w')$,

$$H(w') - I_1(w') = I_2(w') - I^{aux}(w')$$

Thus we prove a polynomial bound for $H(w') - I_1(w')$ on $\Re(w') = \frac{11}{18} + \epsilon$.

Proof. Recall that $H(w')$ is holomorphic in the strip $-\epsilon < \Re(w') < 1+\epsilon$. Also recall

$$I_1(w') \ll Z(\beta') < \infty$$

So $I_1(w')$ is holomorphic throughout the strip. Thus $H(w') - I_1(w')$ is also holomorphic throughout the strip. Recall

$$I_2(w') = C'[Z(w') - Z_1(w')]$$

For $\Re(w') = 1+\epsilon$, since $I^{aux}(w') \ll (q^N)^\gamma$, for $\gamma > 0$, $Z(w') = O(1)$ and $Z_1(w')$ already has polynomial growth in q^N, we conclude that

$$H(w') - I_1(w') = I_2(w') - I^{aux}(w')$$

has polynomial growth in q^N for $\Re(w') = 1+\epsilon$.

Now assume $\Re(w') = -\epsilon$.

$$H(w') - I_1(w') = I(w') - I^{aux}(w') - I_1(w')$$

Again, $I^{aux}(w')$ has polynomial growth for $\Re(w') = -\epsilon$, and $I_1(w') \ll Z(\beta')$. The spectral expansion of $I(w')$ and $I_1(w')$ shows that $I(w')$ and $I_1(w')$ also have polynomial growth for $\Re(w') = -\epsilon$. Thus $H(w') - I_1(w')$ has polynomial growth in q^N for $\Re(w') = -\epsilon$.

□

The Phragmen-Lindelöf theorem implies $I_2(w') - I^{aux}(w')$ has polynomial growth in q^N in $\frac{11}{18} + \epsilon \le \Re(w') \le 1 + \epsilon$. Hence, so has $I_2(w')$. Thus, $Z(w')$ has polynomial growth in q^N in $\frac{11}{18} + \epsilon \le \Re(w') \le 1 + \epsilon$.

6.1 SUBCONVEXITY BOUNDS

Fix a non-archimedean place v_1, take $1 < \beta' < 2$, and fix $0 < t < 1$ in the non-decoupled integral at the archimedean places. $Z(w')$ has analytic continuation to $\Re(w') > \frac{11}{18}$ with a pole of order 1 at $w' = 1$, and has polynomial growth on every vertical strip inside $\frac{11}{18} + \epsilon \le \Re(w') \le 1 + \epsilon$. Choose $\frac{11}{18} < \delta_0 < 1$. From section 4 in [Diaconu-Garrett 2008], for $\delta_0 \le \Re(w') \le 1 + \epsilon$, by Phragmen-Lindelöf, $Z(\delta_0 + i\eta)$ has polynomial growth of exponent less than $\frac{1}{2}$. Consider the rectangle R with vertices at $\delta_0 - iS$, $\beta' - iS$, $\beta' + iS$, $\delta_0 + iS$. Recall Perron's formula: for $\beta' > 1$,

$$\frac{1}{2\pi i}\int_{\beta'-iS}^{\beta'+iS} \frac{x^w}{w}\, dw = \begin{cases} 1 & (\text{for } x > 1) \\ 0 & (\text{for } x < 1) \end{cases} + x^{\beta'} O_{\beta'}(\min\{1, \frac{1}{S|\log x|}\})$$

Applying Perron's formula to the integral

$$\frac{1}{2\pi i}\int_{\beta'-iS}^{\beta'+iS} \frac{Z(w')\, x^{w'}}{w'}\, dw'$$

gives

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{\beta'-iS}^{\beta'+iS} \frac{Z(w')\, x^{w'}}{w'}\, dw' \\ &= \tfrac{1}{2\pi i}\sum_{\chi}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot \Big(\int_{\beta-iS}^{\beta+iS} \frac{(x/q^N)^{w'}}{w'}\, dw'\Big) \times \\ &\qquad\qquad \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt \\ &= \sum_{\chi: q^N \le x}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot 1 \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt \;+\; E(x,S) \end{aligned}$$

where the error term $E(x,S)$ is

$$E(x,S) \ll \sum_{\chi}\int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot (\frac{x}{q^N})^{\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi) \cdot \min\{1, \frac{1}{S|\log(\frac{x}{q^N})|}\}\, dt$$

Theorem 6.1. $\lim_{S\to\infty} E(x,S) = 0, \quad \text{for } x > 0.$

Proof. We first show that

$$\lim_{S\to\infty}\int_{\delta_0+iS}^{\beta'+iS} \frac{Z(w')\, x^{w'}}{w'}\, dw' = 0 \quad \text{and} \quad \lim_{S\to\infty}\int_{\delta_0-iS}^{\beta'-iS} \frac{Z(w')\, x^{w'}}{w'}\, dw' = 0$$

Let $w' = \delta + iS$. Then $Z(w') \ll S^m, \quad \text{for } m < \frac{1}{2} \text{ and } |w'| = \sqrt{\delta^2 + S^2} \ll S$. Thus the integrals above approach 0 as $S \to \infty$. Consider the sets:

$$A = \left\{ N : \frac{1}{S|\log(\frac{x}{q^N})|} \leq \frac{1}{\sqrt{S}} \right\}$$
$$B = \left\{ N : \frac{1}{S|\log(\frac{x}{q^N})|} \geq \frac{1}{\sqrt{S}} \right\}$$

On A,

$$E(x,S) \ll \frac{1}{\sqrt{S}} \sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (\frac{x}{q^N})^{\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi)\, dt$$

$$= \frac{x^{\beta'}}{\sqrt{S}} \sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi)\, dt$$

$$= \frac{x^{\beta'}}{\sqrt{S}} Z(\beta') \text{ where } Z(\beta') \text{ is independent of } S$$

So $\lim_{S\to\infty} E(x,S) = 0$. On B, $\mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi)$ can be estimated by the analytic conductor:

$$Q(\chi,t) = \prod_{v \approx \mathbb{R}} (1 + |t + t_v|) \cdot \prod_{v \approx \mathbb{C}} (1 + \ell_v^2 + 4(t+t_v)^2)$$

Break $E(x,S)$ into two sums, over $q^N \leq \log S$ and $q^N \geq \log S$. Since $Z(w')$ converges absolutely for $\Re(w') > 1$, the second sum over $q^N \geq \log S$ approaches 0. So consider

$$\sum_{\chi : q^N \leq \log S} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot (\frac{x}{q^N})^{\beta'} \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi) \cdot \min\{1, \frac{1}{S|\log(\frac{x}{q^N})|}\}\, dt$$

Now in B,

$$\sum_{q^N \leq \log S} \int_{-\infty}^{\infty} 1 \ll (\log S)^k, \;\; k > 0$$

The convexity bound in the depth aspect gives

$$L(\frac{1}{2} + it, f \otimes \chi) \ll (q^N)^{\frac{1}{2}} \leq (\log S)^{\frac{1}{2}}$$

Fix $\chi = 1$ and $0 < t < 1$ for $v|\infty$. Then $\mathcal{K}_\infty(\frac{1}{2} + it, 0, \beta', \chi) \ll 1$. Also

$$\frac{1}{S|\log(\frac{x}{q^N})|} \geq \frac{1}{\sqrt{S}} \Longrightarrow \frac{1}{\sqrt{S}|\log(\frac{x}{q^N})|} \geq 1 \Longrightarrow xe^{-\frac{1}{\sqrt{S}}} \leq q^N \leq xe^{\frac{1}{\sqrt{S}}}$$

This restricts N to a set of measure $\ll \frac{1}{\sqrt{S}}$. So in the second case $\lim_{S\to\infty} E(x,S) = 0$. □

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_R \frac{Z(w')\,x^{w'}}{w'}\,dw' = xP(\log x)$$

where $P(\log x)$ is a polynomial in $\log x$. So

$$\frac{1}{2\pi i}\int_R \frac{Z(w')\,x^{w'}}{w'}\,dw' = \frac{1}{2\pi i}\int_{\beta'-iS}^{\beta'+iS} \frac{Z(w')\,x^{w'}}{w'}\,dw' - \frac{1}{2\pi i}\int_{\delta_0-iS}^{\delta_0+iS} \frac{Z(w')\,x^{w'}}{w'}\,dw'$$

$$= xP(\log x)$$

Perron's formula showed that

$$\frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} \frac{Z(w')\,x^{w'}}{w'}\,dw' = \sum_{q^N\le x}\int_{-\infty}^{\infty} |L(\tfrac{1}{2}+it, f\otimes\chi)|^2\cdot\mathcal{K}_\infty(\tfrac{1}{2}+it, 0, \beta', \chi)\,dt$$

Thus as $S\to\infty$,

$$\sum_{q^N\le x}\int_{-\infty}^{\infty} |L(\tfrac{1}{2}+it, f\otimes\chi)|^2\cdot\mathcal{K}_\infty(\tfrac{1}{2}+it, 0, \beta', \chi)\,dt = xP(\log x) + \frac{1}{2\pi i}\int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w')\,x^{w'}}{w'}\,dw'$$

Theorem 6.2.

$$\frac{1}{2\pi i}\int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w')\,x^{w'}}{w'}\,dw' \ll x^{\frac{2\delta_0+1}{3}}\cdot\log x \qquad (\frac{11}{18}<\delta_0<1)$$

Proof. By the choice of δ_0, with $w'=\delta_0+i\eta$, $\dfrac{Z(\delta_0+i\eta)}{\delta_0+i\eta}$ is a square integrable function on $\mathbb{R}$. Let

$$E(x) = \frac{1}{2\pi i}\int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w')\,x^{w'}}{w'}\,dw'$$

Lemma 6.3 $\int_0^x |E(t)|^2\,dt \ll x^{2\delta_0+1}$

Proof. Let $x=e^{-2\pi u}$ and again $w'=\delta+i\eta$. Then

$$E(e^{-2\pi u}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Z(\delta_0+i\eta)}{\delta_0+i\eta}\cdot e^{-2\pi u(\delta_0+i\eta)}\cdot i\,d\eta$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\pi iu\eta}\cdot f(\eta)\cdot e^{-2\pi u\delta_0}\,d\eta \quad (\text{where}\quad f(\eta)=\frac{Z(\delta_0+i\eta)}{\delta_0+i\eta})$$

Now

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(\eta)\,e^{-2\pi i\eta u}\,d\eta$$

Thus

$$e^{2\pi u\delta_0}\,\cdot\,2\pi\,\cdot\,E(e^{-2\pi u})=\hat{f}(u)$$

Using Plancherel's theorem:

$$\int_{-\infty}^{\infty}|\hat{f}(u)|^2\,du=\int_{-\infty}^{\infty}|f(\eta)|^2\,d\eta\ll 1$$

So

$$1\gg 4\pi^2\int_{-\infty}^{\infty}|e^{2\pi u\delta_0}\,\cdot\,E(e^{-2\pi u})|^2\,du$$

Replace $e^{-2\pi u}$ by y to get

$$1\gg \tfrac{4\pi^2}{2\pi}\int_0^\infty y^{-2\delta_0}\,\cdot\,|E(y)|^2\,\tfrac{dy}{y}\;=\;2\pi\int_0^\infty y^{-(2\delta_0+1)}\,\cdot\,|E(y)|^2\,dy$$

$$\geq \int_0^x y^{-(2\delta_0+1)}\,\cdot\,|E(y)|^2\,dy\;\geq x^{-(2\delta_0+1)}\int_0^x|E(y)|^2\,dy\qquad\text{for }\;0\leq y\leq x$$

Thus

$$\int_0^x|E(y)|^2\,dy\ll x^{2\delta_0+1},\qquad 0<\delta_0<1$$

□

We now prove Theorem 6.2, namely, that

$$E(x)\ll x^{\frac{2\delta_0+1}{3}}\,\cdot\,\log x$$

First note that $\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)$ is positive. For $x\leq y$

$$\{N:q^N\leq x\}\subseteq\{N:q^N\leq y\}$$

Again,

$$E(x)=\sum_{q^N\leq x}\int_{-\infty}^{\infty}|L(\frac{1}{2}+it,f\otimes\chi)|^2\,\cdot\,\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)\,dt\;-\;xP(\log x)$$

So

$$E(y)-E(x)=\sum_{q^N\leq y}\int_{-\infty}^{\infty}|L(\frac{1}{2}+it,f\otimes\chi)|^2\,\cdot\,\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)\,dt\,-$$

$$\sum_{q^N\leq x}\int_{-\infty}^{\infty}|L(\frac{1}{2}+it,f\otimes\chi)|^2\cdot\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)\,dt\;-\;(yP(\log y)\;-\;xP(\log x))$$

Since $\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)$ is positive,

$$E(y)-E(x)\geq-(yP(\log y)-xP(\log x))$$

Fix $x \geq 3$

(a) Replace y with $x+u$ for $0 \leq u \leq x$:

$$E(x+u) - E(x) \geq -[(x+u)P\log(x+u) - xP(\log x)]$$
$$\Longrightarrow E(x) \leq E(x+u) + (x+u)P\log(x+u) - xP(\log x)$$

and

$$E(x) \leq E(x+u) + Cu\log x \text{ for some constant } C$$

(b) Replace x with $x-u$ and y with x for $0 \leq u < x$. Then

$$E(x) - E(x-u) \geq -[xP(\log x) - (x-u)P\log(x-u)]$$
$$\Longrightarrow E(x) \geq E(x-u) - Cu\log x$$

Let $0 \leq H \leq x$. Integrate the inequalities over $0 \leq u \leq H$:

$$\int_0^H E(x)\,du \leq \int_0^H (E(x+u)+Cu\log x)\,du \;=\; H\cdot E(x) \leq \int_0^H E(x+u)\,du + \frac{C}{2}H^2 \log x$$

and

$$H \cdot E(x) \geq \int_0^H E(x-u)\,du - \frac{C}{2}H^2 \log x$$

So

$$\int_0^H E(x-u)\,du - \frac{C}{2}H^2 \log x \leq H \cdot E(x) \leq \int_0^H E(x+u)\,du + \frac{C}{2}H^2 \log x$$

Change variables and replace $\frac{C}{2}$ by C to get

$$\frac{1}{H}\int_{x-H}^{x} E(t)\,dt \;-\; CH\log x \leq E(x) \leq \frac{1}{H}\int_x^{x+H} E(t)\,dt \;+\; CH\log x$$

For $E(x) \geq 0$, apply the second inequality, otherwise apply the first. So for $E(x) \geq 0$,

$$E(x)^2 \ll \frac{1}{H^2}\left(\int_x^{x+H} E(t)\,dt\right)^2 \;+\; C^2H^2\log^2 x$$

Apply Cauchy-Schwarz:

$$E(x)^2 \ll \frac{1}{H^2}\int_x^{x+H} |E(t)|^2\,dt \cdot \int_x^{x+H} 1\,dt \;+\; H^2\log^2 x$$
$$= \frac{1}{H}\int_x^{x+H} |E(t)|^2\,dt \;+\; H^2\log^2 x \ll \frac{1}{H}\cdot x^{2\delta_0+1} \;+\; H^2\log x$$

since

$$\int_0^x |E(t)|^2\,dt \ll x^{2\delta_0+1} \text{ and } H \leq x$$

We want $\frac{1}{H}\cdot x^{2\delta_0+1} = H^2$, so take $H = x^{\frac{2\delta_0+1}{3}}$. Then

$$E(x) \ll H\log x = x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

□

Recall that the χ-depth-aspect convexity bound is

$$L(\frac{1}{2}+it, f\otimes\chi) \ll q^{N(\frac{d}{2}+\epsilon)}$$

Now use the results above to break convexity by decreasing the exponent of q^N.

Choose H such that

$$x^{\frac{2\delta_0+1}{3}} \ll H \ll x^{\frac{2\delta_0+1}{3}}$$

Let

$$S(x) = \sum_{q^N\le x} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot \mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)\, dt$$
$$= xP(\log x) + O(x^{\frac{2\delta_0+1}{3}} \log x) = xP(\log x) + E(x)$$

Now for $H > 0$, $\{N : q^N \le x\} \subset \{N : q^N \le x+H\}$ and $\mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)$ is *positive*. So for trivial χ,

$$S(x{+}H{+}1){-}S(x) \ge \sum_{x\le q^N\le x+H} \int_{-\infty}^{\infty} |L(\frac{1}{2}{+}it, f{\otimes}\chi)|^2 \cdot \prod_{v|\infty} \mathcal{K}_v(\frac{1}{2}{+}it, 0, \beta', 1)\, dt$$

Now

$$S(x{+}H{+}1){-}S(x) = (x{+}H{+}1)P(\log(x{+}H{+}1)){-}xP(\log x){+}E(x{+}H{+}1){-}E(x)$$

Since $x^{\frac{2\delta_0+1}{3}} \ll H \ll x^{\frac{2\delta_0+1}{3}}$ and $E(x) \ll x^{\frac{2\delta_0+1}{3}}$

$$E(x+H+1) - E(x) \ll x^{\frac{2\delta_0+1}{3}} \log x$$

and

$$(x+H+1)P(\log(x+H+1)) - xP(\log x) \le C(H+1)\log x$$

So

$$S(x+H+1) - S(x) \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

$$\Longrightarrow \sum_{x\le q^N\le x+H} \int_{-\infty}^{\infty} |L(\frac{1}{2}{+}it, f{\otimes}\chi)|^2 \cdot \prod_{v|\infty} \mathcal{K}_v(\frac{1}{2}{+}it, 0, \beta', 1)\, dt \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

Now

$$Q(\chi, t)^{-\beta'} \ll \mathcal{K}_v(\frac{1}{2}+it, 0, w, \chi) \ll Q(\chi, t)^{-\beta'} \quad (\text{for } v|\infty)$$

where

$$Q(\chi, t) = \prod_{v\approx\mathbb{R}} (1+|t+t_v|) \cdot \prod_{v\approx\mathbb{C}} (1+\ell_v^2+4(t+t_v)^2)$$

For trivial χ, $t_v = l_v = 0$. Also recall for $v|\infty$, fix $0 < t < 1$. Then

$$\mathcal{K}_v(\frac{1}{2}+it, 0, \beta', 1) \gg (1)^{-(d-1)\beta'}$$

So

$$x^{\frac{2\delta_0+1}{3}} \log x \gg \sum_{x \le q^N \le x+H} \int_{-\infty}^{\infty} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot \prod_{v|\infty} \mathcal{K}_v(\frac{1}{2}+it, 0, \beta', 1)\, dt$$

$$\gg \sum_{x \le q^N \le x+H} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot (1)^{-(d-1)\beta'}\, dt$$

$$\gg \sum_{x \le q^N \le x+H} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot (q^N)^{-(d-1)\beta'}\, dt$$

$$\ge \sum_{x \le q^N \le x+H} |L(\frac{1}{2}+it, f\otimes\chi)|^2 \cdot (x+H)^{-(d-1)\beta'}$$

So

$$\sum_{x \le q^N \le x+H} |L(\frac{1}{2}+it, f\otimes\chi)|^2\, dt \ll (x+H)^{(d-1)\beta'} \cdot x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

$$\ll x^{d-1+\frac{2\delta_0+1}{3}+\frac{\epsilon}{2}} \cdot \log x \;\text{ where }\; \beta' = 1 + \frac{\epsilon}{2d-2}$$

$$\ll x^{d-1+\frac{2\delta_0+1}{3}+\epsilon}$$

A standard argument, analogous to that in [Good 1982], then gives the pointwise bound

$$L(\frac{1}{2}+it, f\otimes\chi) \ll (q^N)^{\frac{d-1}{2}+\frac{2\delta_0+1}{6}+\epsilon} \ll (q^N)^{\frac{d-1+\vartheta}{2}+\epsilon} \;\text{ for }\; \vartheta < 1$$

.

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