The Riemann Hypothesis

- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ (for $\Re(s) > 1$)

- Bernard Riemann (1859) conjectured that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.

- Open problem - trillion zeros located on the line (Odlyzko)

- David Hilbert (1900): 8^{th} of the 23 problems

- Clay Mathematics Institute (2000): \$1 million dollar prize

RH and primes

Proof of RH will provide insight into the distribution of primes.

- $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$

- Prime number theorem:

$$\pi(x) \sim \frac{x}{\log x}$$

or

$$\lim_{x\to\infty} \pi(x) \cdot \frac{\log x}{x} = 1$$

- Hadamard and de la Vallée Poussin (1896)

- Key step in proof: $\zeta(s)$ has no zeros on $\Re(s) = 1$.

- Showed that

$$\pi(x) = Li(x) + O(xe^{-a\sqrt{\log x}})$$

for some positive constant a, where

$$Li(x) = \int_2^x \frac{1}{\log t}\, dt$$

Error term $O(xe^{-a\sqrt{\log x}})$ dependent on zero-free region within critical strip.

- Truth of RH $\Rightarrow$ the smallest possible error estimate in PNT.

- RH equivalent to

$$\pi(x) = Li(x) + O(x^{\frac{1}{2}} \log x)$$

and

$$\pi(x) - Li(x) = O(x^{\frac{1}{2}+\epsilon}) \quad \text{for all} \quad \epsilon \; > 0$$

Grand Riemann Hypothesis

- Automorphic L-function:

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- Dirichlet L-function:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

  χ is a Dirichlet character - a complex-valued function defined for integers with the mutiplicative property: $\chi(ab) = \chi(a)\chi(b)$.

- Grand RH - extension of RH to L-functions.

Lindelöf Hypothesis

- $\zeta(s)$ grows slowly on the critical line.

- E. Lindelöf (1908):

$$\zeta(\frac{1}{2}+it) \ll (|t|^{\epsilon}) \quad \text{(for all } \epsilon > 0)$$

- Convexity or trivial bound:

$$|\zeta(\frac{1}{2}+it)| \ll |t|^{\frac{1}{4}+\epsilon}$$

- Breaking convexity - Weyl (1921):

$$|\zeta(\frac{1}{2}+it)| \ll |t|^{\frac{1}{6}+\epsilon}$$

- Burgess (1962):

$$|L(\frac{1}{2}+it,\chi)| \ll |t|^{\frac{3}{16}+\epsilon} \ \ (\chi \text{ a Dirichlet character})$$

Moments

- LH is equivalent to:

  $$I_k(T) = \int_{-T}^{T} |\zeta(\frac{1}{2}+it)|^k \, dt \sim T^{1+\epsilon} \quad (\text{for } k = 2, 4, 6, \ldots)$$

  k^{th} integral moment of $\zeta(s)$ on critical line.

- Grand LH

- Second moment - Hardy-Littlewood(1918):

  $$\int_0^T |\zeta(\frac{1}{2}+it)|^2 \, dt \sim T \log T$$

- Fourth moment - Ingham(1926):

  $$\int_0^T |\zeta(\frac{1}{2}+it)|^4 \, dt \sim \frac{1}{2\pi^2} \cdot T(\log T)^4$$

- Diaconu-Garrett (2006): asymptotics with error-term for second moment of families of $GL(2)$ L-functions over arbitrary number field. Broke convexity in t-aspect.

- My research - change character χ and break convexity in χ-aspect.

Applications

- Equidistribution problems
- Quantum unique ergodicity