The Riemann Hypothesis

• \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) (for \( \Re(s) > 1 \))

• Bernard Riemann (1859) conjectured that all non-trivial zeros of \( \zeta(s) \) lie on the critical line \( \Re(s) = \frac{1}{2} \).

• Open problem - trillion zeros located on the line (Odlyzko)

• David Hilbert (1900): 8\(^{th} \) of the 23 problems

• Clay Mathematics Institute (2000): $1 million dollar prize
RH and primes

Proof of RH will provide insight into the distribution of primes.

• \( \zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \)

• Prime number theorem:
  \[ \pi(x) \sim \frac{x}{\log x} \]
  or
  \[ \lim_{x \to \infty} \pi(x) \cdot \frac{\log x}{x} = 1 \]

• Hadamard and de la Vallée Poussin (1896)

• Key step in proof: \( \zeta(s) \) has no zeros on \( \Re(s) = 1 \).

• Showed that
  \[ \pi(x) = Li(x) + O(xe^{-a\sqrt{\log x}}) \]
  for some positive constant \( a \), where
  \[ Li(x) = \int_2^x \frac{1}{\log t} \, dt \]
Error term $O(xe^{-a\sqrt{\log x}})$ dependent on zero-free region within critical strip.

- Truth of RH $\Rightarrow$ the smallest possible error estimate in PNT.

- RH equivalent to

$$
\pi(x) = Li(x) + O(x^{\frac{1}{2}} \log x)
$$

and

$$
\pi(x) - Li(x) = O(x^{\frac{1}{2}+\epsilon}) \text{ for all } \epsilon > 0
$$
Grand Riemann Hypothesis

- Automorphic $L$-function:
  \[ L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \]

- Dirichlet $L$-function:
  \[ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \]

  $\chi$ is a Dirichlet character - a complex-valued function defined for integers with the multiplicative property: $\chi(ab) = \chi(a)\chi(b)$.

- Grand RH - extension of RH to $L$-functions.
Lindelöf Hypothesis

- \( \zeta(s) \) grows slowly on the critical line.

- E. Lindelöf (1908):
  \[
  \zeta(\frac{1}{2} + it) \ll (|t|^{\epsilon}) \quad (\text{for all } \epsilon > 0)
  \]

- Convexity or trivial bound:
  \[
  |\zeta(\frac{1}{2} + it)| \ll |t|^\frac{1}{4} + \epsilon
  \]

- Breaking convexity - Weyl (1921):
  \[
  |\zeta(\frac{1}{2} + it)| \ll |t|^\frac{1}{6} + \epsilon
  \]

- Burgess (1962):
  \[
  |L(\frac{1}{2} + it, \chi)| \ll |t|^\frac{3}{16} + \epsilon \quad (\chi \text{ a Dirichlet character})
  \]
Moments

• LH is equivalent to:
  \[ I_k(T) = \int_{-T}^{T} |\zeta(\frac{1}{2} + it)|^k \, dt \sim T^{1+\epsilon} \text{ (for } k = 2, 4, 6, \ldots) \]
  
  \[ k^{th} \text{ integral moment of } \zeta(s) \text{ on critical line.} \]

• Grand LH

• Second moment - Hardy-Littlewood(1918):
  \[ \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt \sim T \log T \]

• Fourth moment - Ingham(1926):
  \[ \int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \sim \frac{1}{2\pi^2} \cdot T(\log T)^4 \]

• Diaconu-Garrett (2006): asymptotics with error-term for second moment of families of \( GL(2) \) \( L \)-functions over arbitrary number field. Broke convexity in \( t \)-aspect.

• My research - change character \( \chi \) and break convexity in \( \chi \)-aspect.
Applications

- Equidistribution problems

- Quantum unique ergodicity