Target Tracking Error

A utility-based model for active management
Background

• Active Asset Management
• Net Asset Value
• Relative Performance
• Tracking Error
• Information Ratio
Objective

Inform the control of the deployment of tracking error to maximize investor utility
Performance Process

\[ d\Pi_t = \lambda(v) \cdot v \, dt + v \, dW_t \]

- \( v \) = tracking error is the controlling variable
- \( \lambda \) = information ratio is a function of the tracking error
- **caveat:** normal increments
Dilution from Scale

\[
\frac{d\lambda}{dv} = -\kappa \cdot \frac{\lambda}{v}
\]

- \( \kappa \) is measure of elasticity
  - \( \kappa = 0 \) scales perfectly
  - \( \kappa = 1 \) does not scale at all
Investor Utility

\[ U = \log \Phi \left( \frac{\Pi_T - \mu \cdot T}{\sigma \cdot \sqrt{T}} \right) \]

- Logarithm of the rank quantile
- Depends on
  - horizon
  - competition
Optimality Condition

$$\nu_t = \arg \max_{\nu} E_t U$$
Myopic Optimality

\[ \lim_{t \to T} v_t = \frac{\sigma}{\Phi' \circ \Phi^{-1}(q_t) + \Phi^{-1}(q_t)} \]

\[ \kappa + \frac{q_t}{\lambda \cdot \sqrt{T}} \]

- analytic solution – big benefit!
- does not suffer from fiduciary criticism of arbitrary fixed horizon
Simulation
Simulation

Bottom Quarter: 17%

Top Half: 54 %

annual quantile (0=worst, 1=best)
Transaction Costs

\[ \lambda \cdot \sqrt{T} = \gamma \cdot \sqrt{N} \]
\[ \gamma \approx 2 \cdot p - 1 - \rho \]

- \( \gamma \) = information coefficient
- \( N \) = breadth \( \times \) \( T \)
- \( \rho \) = average round-trip transaction cost
- \( p \) = batting average
Questions?
Implied Track Record

A bayesian estimator for the information ratio
Background

• The “Information ratio” or “Sharpe ratio” is the market price of risk for asset allocation or active fund management

• It has the form of a t-statistic (average ÷ standard deviation)

• It is essentially impossible to observe using unbiased estimators
Objective

Present an information ratio estimator with a coherent trade-off between bias and standard error
Hidden Mixture Model

\[ X_i | \mu, \sigma \overset{iid}{\sim} N(\mu, \sigma^2) \]
\[ \mu | \sigma \sim N\left(\mu_0, \frac{\sigma^2}{\lambda_0}\right) \]
\[ \frac{1}{\sigma^2} \sim \Gamma(\alpha_0, \beta_0) \]

- return observations are independent normals
- four hidden parameters
Parameter Updates

\[
\mu_n = \frac{\lambda_0 \cdot \mu_0}{\lambda_0 + n} + \frac{1}{\lambda_0 + n} \cdot \sum_{i=1}^{n} x_i
\]

\[
\lambda_n = \lambda_0 + n
\]

\[
\alpha_n = \alpha_0 + \frac{1}{2} \cdot n
\]

\[
\beta_n = \beta_0 + \frac{1}{2} \cdot \left\{ \sum_{i=1}^{n} x_i^2 - \frac{1}{\lambda_0 + n} \cdot \left( \sum_{i=1}^{n} x_i \right)^2 + \frac{\lambda_0 \cdot \mu_0}{\lambda_0 + n} \cdot \left( n \cdot \mu_0 - 2 \cdot \sum_{i=1}^{n} x_i \right) \right\}
\]

- Model is invariant under sampling
Process Version

\[
\mu_T = \frac{\mu_0 \cdot \lambda_0 + x_T - x_0}{\lambda_0 + T}
\]

\[
\lambda_T = \lambda_0 + T
\]

\[
\alpha_T = \alpha_0 + \frac{1}{2} \cdot \frac{T}{\Delta t}
\]

\[
\beta_T = \beta_0 + \frac{1}{2} \cdot \left\{ \frac{T}{\Delta t} \sum_{i=1}^{T/\Delta t} \frac{\Delta x_{t_i}^2}{\Delta t} + \mu_0^2 \cdot \lambda_0 - \left( \frac{\mu_0 \cdot \lambda_0 + x_T - x_0}{\lambda_0 + T} \right)^2 \right\}
\]

- Process is sampled discretely
  - This can also be done for aperiodic sampling
Posterior Moments

\[
E\left[ \frac{\mu}{\sigma} \middle| x_{t_i} \right] = \mu_T \cdot \frac{\sqrt{\alpha_T}}{\sqrt{\beta_T}} \cdot \frac{\Gamma(\alpha_T + \frac{1}{2})}{\Gamma(\alpha_T) \cdot \sqrt{\alpha_T}}
\]

\[
\text{var}\left[ \frac{\mu}{\sigma} \middle| x_{t_i} \right] = \frac{1}{\lambda_T} + E\left[ \frac{\mu}{\sigma} \middle| x_{t_i} \right]^2 \cdot \left[ \left( \frac{\Gamma(\alpha_T) \cdot \sqrt{\alpha_T}}{\Gamma(\alpha_T + \frac{1}{2})} \right)^2 - 1 \right]
\]

- Use expected value as the estimator
- Use sqrt variance as the standard error
The Estimation Problem

\[ \text{var} \left[ \frac{\mu}{\sigma} \left\{ x_{t_i} : i = 1, \ldots, \frac{T}{\Delta t} \right\} \right] \geq \frac{1}{\lambda_0 + T} \]

- This does not depend on the sampling frequency
- Unreasonable to assume that population value is that stable!
Solution Characteristics

- Unbiased estimators are too noisy
- Need to inject information to lower the standard error
- Assumptions should be readily understandable
- Solution should be perturbative in the assumptions and unbiased in limit of sample size
“Implied Track Record”

• Assume an implied history
  – given duration
  – given average tracking error
  – given average information ratio

• The analogy is a new manager with an established history elsewhere
Prior Parameters

\[ \lambda_0 = T_0 \]

\[ \alpha_0 = \frac{1}{2} \cdot \frac{T_0}{\Delta t} \]

\[ \beta_0 = E[\sigma|x_0]^2 \cdot \frac{1}{2} \cdot \frac{T_0}{\Delta t} \cdot \left(1 - \frac{\Delta t}{T_0}\right)^2 \cdot \left[ \frac{\Gamma\left(\frac{1}{2} \cdot \frac{T_0}{\Delta t}\right) \cdot \sqrt{\frac{1}{2} \cdot \frac{T_0}{\Delta t}}}{\Gamma\left(\frac{1}{2} \cdot \frac{T_0}{\Delta t} + 1\right)} \right]^2 \]

\[ \mu_0 = E\left[\frac{\mu}{\sigma}|x_0\right] \cdot E[\sigma|x_0] \cdot \left(1 - \frac{\Delta t}{T_0}\right) \cdot \left[ \frac{\Gamma\left(\frac{1}{2} \cdot \frac{T_0}{\Delta t}\right) \cdot \sqrt{\frac{1}{2} \cdot \frac{T_0}{\Delta t}}}{\Gamma\left(\frac{1}{2} \cdot \frac{T_0}{\Delta t} + 1\right)} \right]^2 \]
Questions?