1 Derivation

Consider a smooth curve \( x(t) : \mathbb{R} \rightarrow \mathbb{R}^n \). Let the distance to the center of curvature be \( R(t) \). This means that the angle \( \theta \) between the tangent vectors \( x(t) \) and \( x(t+h) \) for some small \( h \) satisfies

\[
x'(t) \cdot x'(t+h) \approx |x'(t)||x'(t+h)| \cos \theta(t,h) \tag{1}
\]

(a standard result about the scalar product); and, for small angles,

\[
\theta(t,h) \approx \frac{|x(t+h) - x(t)|}{R(t)} \tag{2}
\]

We can use the expansions

\[
x(t+h) \approx x(t) + hx'(t)
\]

\[
x'(t+h) \approx x'(t) + hx''(t)
\]

and

\[
\cos \theta \approx 1 - \frac{1}{2} \theta^2
\]

wherby (2) gives

\[
\cos \theta(t,h) \approx 1 - \frac{1}{2} h^2 \frac{|x'(t)|^2}{R(t)^2} \tag{3}
\]

and (1) gives

\[
\cos \theta(t,h) \approx \frac{|x'(t)|^2 + hx'(t) \cdot x''(t)}{|x'(t)| \sqrt{|x'(t)|^2 + 2hx'(t) \cdot x''(t) + h^2 |x''(t)|^2}}
\]

\[
= \frac{1 + h \frac{x'(t) \cdot x''(t)}{|x'(t)|^2}}{\sqrt{1 + 2h \frac{x'(t) \cdot x''(t)}{|x'(t)|^2} + h^2 \frac{|x''(t)|^2}{|x'(t)|^2}}}
\]
To lowest order in $h$, this last result is
\[
\cos \theta(t, h) \approx 1 - \frac{1}{2} h^2 \left( \frac{|x''(t)|^2}{|x'(t)|^2} - \left( \frac{x'(t) \cdot x''(t)}{|x'(t)|^2} \right)^2 \right) \tag{4}
\]

Identifying (3) and (4), the limit $h \to 0$ gives the result
\[
R(t) = \sqrt{\frac{|x'(t)|^3}{|x'(t)|^2 |x''(t)|^2 - (x'(t) \cdot x''(t))^2}} \tag{5}
\]

## 2 Invariance

This is an intrinsic property of the curve, in the sense that it invariant under a smooth, monotonic transformation of the parameter. To see this, let $t = \phi(s)$ and let $y(s) = x(\phi(s))$. Since
\[
y'(s) = x'(t) \phi'(s) \\
y''(s) = x''(t) \phi'(s)^2 + x'(t) \phi''(s)
\]

It is straight-forward to show that
\[
(y'(s) \cdot y'(s)) (y''(s) \cdot y''(s)) - (y'(s) \cdot y''(s))^2 = \left[ (x'(t) \cdot x'(t)) (x''(t) \cdot x''(t)) - (x'(t) \cdot x''(t))^2 \right] \phi'(s)^6
\]
so
\[
R_y(s) = R_x(t)
\]

The natural arc-length parameterization has $|y'_0(s)| = 1$, whereby
\[
R_{y_0}(s) = \frac{1}{\sqrt{(y''_0(s) \cdot y'_0(s))^2 - (y'_0(s) \cdot y''_0(s))^2}}
\]

### 2.1 Co-ordinate free

Note that the denominator of (5) can be interpreted as
\[
\sqrt{(x' \cdot x') (x'' \cdot x'') - (x' \cdot x'')^2} = \sqrt{\begin{vmatrix} x' \cdot x' & x' \cdot x'' \\ x' \cdot x'' & x'' \cdot x' \\ \end{vmatrix}}
\]
\[
= \sqrt{\begin{vmatrix} x' & x'' \\ x' & x'' \\ \end{vmatrix}}
\]

Since $\det(AB) = (\det A)(\det B)$ and $\det A^T = \det A$ for square matrices $A$ and $B$, it is suggestive to extend the definition of determinant to non-square matrices by
\[
(\det C)^2 \triangleq \sqrt{\det C^T C}
\]
So, in this coordinate-free notation,

\[ R = \frac{|x'|^3}{|\det (x', x'')|} \]

### 3 Cartesian Graph

We can interpret a cartesian graph as a plane curve by the map

\[ x(t) = (t, f(t))^T \]

with curvature

\[ R(t) = \frac{(1 + f'(t)^2)^{3/2}}{|f''(t)|} \]

or in Leibniz’s notation,

\[ R = \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{3/2} \]

\[ \frac{d^2y}{dx^2} \]