Equilibrium allocation under exponential utility & normal markets

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This argument, working notes really, is based on the Black-Litterman analysis as presented in Meucci’s text on risk and asset allocation which is the basis for a course module I teach for the University of Minnesota Center for Financial and Actuarial Mathematics.

Model

For a portfolio \( \alpha_0 + \alpha \) with net asset value

\[
w = \alpha_0 + \alpha^\top p
\]

the gain/loss over \( \tau \) years would be

\[
\Psi = \alpha_0 r \tau + \alpha^\top M
\]

where the market vector is

\[
M = P - p
\]

with \( P \) the random variable for asset prices, including any cashflows, \( \tau > 0 \) years in the future and \( r \) the (simple-interest) return on cash.

Let us assume that the market vector is normal,

\[
M \sim \mathcal{N}(\mu \tau, \Sigma \tau)
\]

and that the preferences of the representative agent are described by exponential utility

\[
u(\psi) = -e^{-\psi/\zeta}
\]

with absolute risk aversion \( 1/\zeta > 0 \).

Let us consider the portfolios that satisfy a wealth constraint \( w^* \) and maximize expected utility.

\[
E u(\Psi) = -e^{-\alpha_0 r \tau} E e^{-\alpha^\top M}
= -e^{-w^* \alpha^\top (r \tau - \mu \tau \Sigma^{-1} \mu) + \frac{1}{2} \alpha^\top \Sigma \alpha}
\]

So an optimal portfolio satisfies

\[
\alpha^* \in \arg \max_{\alpha} \alpha^\top (\mu - rp) - \frac{1}{2\zeta} \alpha^\top \Sigma \alpha
\]
If the covariance is positive-definite, $\Sigma > 0$ (which it would not be if cash were included in the market vector), the first-order condition on the optimal portfolio is

$$0 = \mu - rp - \frac{1}{\zeta} \Sigma \alpha^*$$

Notice that

$$E \Psi^* = w^* r\tau + \frac{1}{\zeta} \text{var} \Psi^*$$

and more generally that

$$E \Psi = w\tau + \frac{1}{\zeta} \text{cov} (\Psi, \Psi^*)$$

This is more recognizable as

$$E \frac{\Psi}{w\tau} = r + \frac{\text{cov} (\Psi, \Psi^*)}{\text{var} \Psi^*} \left( E \frac{\Psi^*}{w^* \tau} - r \right)$$

where the coefficient is akin to “beta” in the capital asset pricing model.

Consider a portfolio consisting of a single share of the $i$-th stock.

$$\frac{\Psi}{w} = \frac{P_i}{p_i} - 1$$

Hence

$$E P_i = p_i (1 + \bar{R}_i \tau) + \lambda \text{cor} (P_i, \Psi^*) \sqrt{\text{var} P_i}$$

where

$$\lambda \triangleq \frac{\sqrt{\alpha^*} \Sigma \alpha^*}{\zeta}$$

with dimensions $y_r^{-1/2}$ is termed the “market price of risk” and notably depends on neither the asset nor the investment horizon.

In particular, the expected value of the (simple) return on the $i$-th asset is

$$\bar{R}_i \triangleq r + \lambda \text{cor} (P_i, \Psi^*) \sqrt{\frac{\text{var} P_i}{P_i^2 \tau}}$$

whereby

$$E P_i = p_i (1 + \bar{R}_i \tau)$$