A General Approach to Portfolio Risk Measurement

by

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Portfolio Risk Measurement

- The economic risk / return trade-off
  - Theoretical & Practice challenges
- Measurement vs. Management
  - Control vs. Trading
- Standards & Regulations
  - Generally Accepted Principals, disclosure
Risk Classifications

- **Bancassurance**: Banking & Insurance
  - Banking
    - Market Risk
    - Credit Risk
    - Liquidity Risk
    - Operational Risk
  - Insurance
    - Property / Casualty …
Market Risk

- On-balance sheet
  - Assets
  - Liabilities

- Off-balance sheet
  - Reserves
  - Translations
  - Hedges
Brownian Motion

- Continuous
- Finite quadratic variation
- Normal increments

\[ dz \equiv dB(t) \quad dz \cdot dz = dt \]

\[ B(t + s) - B(t) \sim N(0, s^+) \]
Market Factors

- Factors may or may not be prices
- Integrands are general non-stochastic functions of factor levels and time

\[ dx^i = M^i(t, x^k) \, dt + L^i_j(t, x^k) \, dz^j \]
\[ dx^i \cdot dx_j = L^i_k \cdot L^k_j \, dt \equiv \Sigma_j dt \]
Example From Cholesky Decomposition

- Two-factor correlated geometric Brownian motion

\[
\begin{pmatrix}
  M^1 \\
  M^2
\end{pmatrix} =
\begin{pmatrix}
  x^1 \cdot \mu^1 \\
  x^2 \cdot \mu^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  L^1_1 & L^1_2 \\
  L^2_1 & L^2_2
\end{pmatrix} =
\begin{pmatrix}
  x^1 \cdot \sigma^1 & 0 \\
  x^2 \cdot \sigma^2 \cdot \rho & x^2 \cdot \sigma^2 \cdot \sqrt{1 - \rho^2}
\end{pmatrix}
\]
Portfolio

- Value depends on factor levels and time
- Assumption: no cash flows, differentiable

\[ \Pi(t, x^i) \]
Portfolio Process

- From Itô’s lemma, we can write down the process for the portfolio value

\[ d\Pi = \frac{\partial \Pi}{\partial t} \, dt + \frac{\partial \Pi}{\partial x^i} \, dx^i + \frac{1}{2} \cdot \frac{\partial^2 \Pi}{\partial x^i \partial x^j} \, dx^i \cdot dx^j \]
Portfolio Partials

- Observable parameters
  - Time decay, position, & convexity

\[ \frac{\partial \Pi}{\partial t} \equiv \Theta \]
\[ \frac{\partial \Pi}{\partial x^i} \equiv \Delta_i \]
\[ \frac{\partial^2 \Pi}{\partial x^i \partial x_j} \equiv \Gamma^j_i \]

\[ d\Pi = \Theta \, dt + \Delta_i \cdot \left( M^i \, dt + L^i_j \, dz^j \right) + \frac{1}{2} \Gamma^j_i \cdot \Sigma^i_j \, dt \]
Portfolio Arbitrage

- **Ansatz**: There exists a hedging portfolio
  - Same delta
  - Different theta and gamma (could be zero)

\[
d\Pi' = \Theta' dt + \Delta_i \cdot \left(M^i dt + L^i_j d\zeta^j\right) + \frac{1}{2} \cdot \Gamma_i^j \cdot \Sigma^i_j dt
\]
Portfolio Arbitrage

- The hedged portfolio is risk-free

\[ d(\Pi' - \Pi) = \left( \Theta' + \frac{1}{2} \Gamma_i^j \cdot \Sigma_j^i \right) dt - \left( \Theta + \frac{1}{2} \Gamma_i^j \cdot \Sigma_j^i \right) dt \]
\[ = (\Pi' - \Pi) \cdot r \, dt \]
Factor Price

- The resulting operator is proportional to delta
  - Implicit function theorem for linear systems
- Defines an arbitrage constraint

\[
\Pi \cdot r - \Theta - \frac{1}{2} \Gamma_i^j \cdot \Sigma_j^i \equiv \Delta_k \cdot S^k \cdot r
\]
Portfolio Process

- Insert the arbitrage constraint into the portfolio process to eliminate gamma & theta

\[ d\Pi = \left\{ \Pi \cdot r + \Delta_i \cdot (M^i - S^i \cdot r) \right\} dt + \Delta_i \cdot L^i_j \; dz^j \]
We are led to define a new portfolio consisting of fully-leveraged positions:

\[\tilde{M}^i \equiv M^i - S^i \cdot r\]
\[d\tilde{\Pi} \equiv d\Pi - \Pi \cdot r \, dt\]
\[d\tilde{\Pi} = \Delta_i \cdot \tilde{M}^i \, dt + \Delta_i \cdot L^i_j \, dz^j\]
Risk As Quadratic Variation

- Instantaneous variance of portfolio profit/loss

\[ d\Pi \cdot d\Pi = \Delta_i \cdot L^i_k \cdot L^k_j \cdot \Delta^j \, dt \]
\[ = \Delta_i \cdot \sum_j^i \cdot \Delta^j \, dt \]
\[ \equiv V^2 \, dt \]
Marginal Risk

- The incremental risk from a given position
  - The basis for decision making
- Additive decomposition of total risk

\[
N^i \equiv \frac{\partial V}{\partial \Delta_i} = \frac{\Sigma^i_j \cdot \Delta^j}{V}
\]

\[
V \equiv \sqrt{\Delta_i \cdot \Sigma^i_j \cdot \Delta^j} = \Delta_i \cdot N^i
\]
Marginal Risk

- Project portfolio onto a single stochastic component

\[ d\tilde{\Pi} = \Delta_i \cdot \tilde{M}^i \ dt + \Delta_i \cdot N^i \ dz \]
Efficiency

- The marginal return should be proportional to the marginal risk
  - Maximum return for a given level of risk

\[
\tilde{N}^i \equiv \lambda \cdot \tilde{M}^i
\]

\[
\lambda \equiv \sqrt{\tilde{M}_j \cdot T^{i}_{j} \cdot \tilde{M}^i}
\]

\[
T \equiv \Sigma^{-1}
\]
Efficiency

- This has various interpretations
  - Unique market price of risk
  - Slope of the Capital Market Line

\[ d\tilde{\Pi}_{\text{eff}} = V \cdot \lambda \, dt + V \, dz \]

\[ \lambda = \frac{\mu - r}{\sigma} \]
Finite Increments

- Over a finite time horizon, the portfolio profit/loss is given by a stochastic integral

\[
\delta \Pi \equiv \int_0^{\delta t} d\Pi \\
= \int_0^{\delta t} \left( \Pi \cdot r + \Delta_i \cdot \tilde{M}^i \right) dt + \int_0^{\delta t} \Delta_i \cdot L^i_j \, dz^j
\]
FiniteIncrements

- The stochastic term generally dominates the non-stochastic term for short time intervals
- Allow it to vary with respect to factor levels
  - e.g. Geometric Brownian motion, trading patterns

\[ \Delta_i(x^k) \cdot L^{i\ j}(x^k) \]
Linear Approximation

- First-order Taylor’s expansion

\[
\Delta_i \cdot L^i_j \approx \Delta_i \bigg|_0 \cdot L^i_j \bigg|_0 + \left( \frac{\partial \Delta_i}{\partial x_k} \bigg|_0 \cdot L^i_j \bigg|_0 + \Delta_i \bigg|_0 \cdot \frac{\partial L^i_j}{\partial x_k} \bigg|_0 \right) \cdot \frac{\partial x_k}{\partial z_l} \bigg|_0 \cdot z_l
\]

\[
= \Delta_i \bigg|_0 \cdot L^i_j \bigg|_0 + \left( \Gamma^k_i \bigg|_0 \cdot L^i_j \bigg|_0 + \Delta_i \bigg|_0 \cdot L^{i\cdot k}_j \bigg|_0 \right) \cdot L^l_k \bigg|_0 \cdot z_l
\]
Stochastic Integration Results

- Integrate one Brownian motion w.r.t. another

\[
\begin{align*}
\int_0^{\delta t} dz^j &= z^j \cdot \sqrt{\delta t} , \quad z^j \sim \text{iid } N(0,1) \\
\int_0^{\delta t} z_l \, dz^j &= \frac{1}{2} \left( z_l \cdot z^j - \delta^j_l \right) \cdot \delta t
\end{align*}
\]
Covariant Adjustments

- It is a useful simplification to fold in the factor convexities at this point

\[
\tilde{\Gamma}_i^k \equiv \Gamma_i^k + \Delta_i \cdot \frac{L^i_j,^k}{L^i_j}
\]

\[
\tilde{\Theta} \equiv \Theta - \frac{1}{2} \cdot \left(\tilde{\Gamma}_i^k - \Gamma_i^k\right) \cdot \Sigma_k
\]
Approximate Result

- The portfolio profit/loss in the finite case under the linear approximation is of the form below

\[ \delta \Pi \approx \delta \Pi' = T \cdot \delta t + \left( D_j \cdot z^j \right) \cdot \sqrt{\delta t} + \frac{1}{2} \left( z_l \cdot G^l_{j} \cdot z^j \right) \cdot \delta t \]
Coefficients

- Combine results from above to get the following

\[ T \equiv \tilde{\Theta}_0 + \Delta_i \cdot M^i \]

\[ D_j \equiv \Delta_i \cdot L^i_j \]

\[ G^l_{j} \equiv L^l_k \cdot \tilde{\Gamma}^k_i \cdot L^i_j \]
Quadratic Form

- Since this is part of a quadratic form,
  - Symmetrize
  - Diagonalize
    - Eigenvalues are real; Transformation is orthogonal

\[
G_l^j \equiv \frac{1}{2} \cdot \left( G_l^j + G_j^l \right) \\
\equiv O_j^k \cdot \tilde{G}_i^k \cdot O_l^i
\]
Quadratic Form

- Re-express along Eigenvectors
  - Still independent standard Normal variants

\[
\tilde{z}^i \equiv O^i_k \cdot z^k, \quad \tilde{z}^i \sim \text{iid } N(0,1)
\]
\[
\tilde{D}_i \equiv D_j \cdot O^j_i
\]
Approximate Result

- The result is a constant plus a sum of independent non-central chi-squared random variables

\[
\delta \Pi' = T \cdot \delta t + \left( \tilde{D}_i \cdot \tilde{z}^i \right) \cdot \sqrt{\delta t} + \frac{1}{2} \left( \tilde{G}_i \cdot \tilde{z}_i \cdot \tilde{z}^i \right) \cdot \delta t
\]

\[
= \left( T - \frac{\tilde{D}_i^2}{2 \cdot \tilde{G}_i \cdot \delta t} \right) \cdot \delta t + \frac{1}{2} \tilde{G}_i \cdot \left( \tilde{z}^i + \frac{\tilde{D}_i}{\tilde{G}_i \cdot \sqrt{\delta t}} \right)^2 \cdot \delta t
\]
Value-at-Risk

- The standard measure for market risk
  - Depends on time horizon & confidence level

\[ \Pr \{ \delta \Pi(\delta t) < -P \} \equiv \alpha \]
Value-at-Risk

- Standard parameters (Basle Accord)
  - Two-week time horizon
  - Three normal standard-deviation confidence level

\[
\delta t = \frac{14}{365.25} = 0.0383\ldots
\]

\[
\alpha = \Phi(-3) = 0.0013\ldots
\]
Value-at-Risk

- Different approaches to solution
  - Central Limit Theorem
  - Monte Carlo simulation
  - Quadrature over conic sections
  - “Linearization”
    - Consistent with simpler approaches
    - Accounts for both sign & magnitude of gamma
Linearization

- Standardize each random variable

\[ Z_i(\varepsilon) = \text{sgn}(\varepsilon) \cdot \frac{\left(\text{sgn}(\tilde{D}_i) \cdot \tilde{z}_i + \varepsilon\right)^2 - (1 + \varepsilon^2)}{\sqrt{2 + 4 \cdot \varepsilon^2}} \]
Linearization

- Define the reciprocal of the non-centrality parameter as the curvature

\[ K^i \equiv \frac{\tilde{G}_i^i}{\tilde{D}_i} \cdot \sqrt{\delta t} \]

\[ \delta \Pi' = \left( T + \frac{1}{2} : \tilde{G}_i^i \right) \cdot \delta t + \sqrt{1 + \frac{1}{2} K_i^2} \cdot |\tilde{D}_i| \cdot Z^i \cdot \sqrt{\delta t} \]
Tailoring

- Scale the standardized variables by the ratio of the tail mass to the normal tail mass

\[
\hat{Z}^i \equiv \frac{\Phi^{-1}(\alpha)}{F_{Z^i}^{-1}(\alpha)} \cdot Z^i
\]
Tailoring

- Or equivalently, scale the delta
- And make one final adjustment to the theta

\[
\hat{D}_i \equiv \left\{ \frac{F_{Z_i}^{-1}(\alpha)}{\Phi^{-1}(\alpha)} \cdot \sqrt{1 + \frac{1}{2} \cdot K_i^2} \right\} \cdot \tilde{D}_i
\]

\[
\hat{T} \equiv T + \frac{1}{2} \cdot \tilde{G}_i^i
\]
Tailoring

99% confidence

delta multiple vs. curvature
Approximate Result

- All of this finally allows us to write the portfolio innovation in a quasi-linear form

\[ \delta \Pi' = \hat{T} \cdot \delta t + \left| \hat{D}_i \right| \cdot \hat{Z}^i \cdot \sqrt{\delta t} \]
Normal Approximation

- Now, we apply a second approximation by replacing the standardized non-central chi-squared random variables by normal random variables

\[ \delta \Pi' \approx \delta \Pi'' \equiv \hat{T} \cdot \delta t + \sqrt{\hat{D}_i \cdot \hat{D}^i} \cdot \sqrt{\delta t} \cdot z \]

with \( z \sim N(0,1) \)
“Linearized” Value-at-Risk

- It is straightforward to solve for the Value-at-Risk in this case
- The drift may be suppressed as a third approximation

\[ P'' = -\Phi^{-1}(\alpha) \cdot \sqrt{\hat{D}_i \cdot \hat{D}^i} \cdot \sqrt{\delta t} - \hat{T} \cdot \delta t \]

\[ P'' \approx P''' \equiv -\Phi^{-1}(\alpha) \cdot \sqrt{\hat{D}_i \cdot \hat{D}^i} \cdot \sqrt{\delta t} \]