Introduction

- chapter 7 is a continuation of chapter 3
- but we will drop Assumption 3.2, independence
- ...and specify (risk-neutral) stochastic dynamics
- since the building block securities can be expressed as

\[
B(t, T) = E^Q \left[ e^{-\int_t^T r(s) \, ds} \bigg| \mathcal{F}_t \right]
\]

\[
\bar{B}(t, T) = E^Q \left[ e^{-\int_t^T r(s) + \lambda(s) \, ds} \bigg| \mathcal{F}_t \right]
\]

\[
e(t, T) = E^Q \left[ \lambda(T) \cdot e^{-\int_t^T r(s) + \lambda(s) \, ds} \bigg| \mathcal{F}_t \right]
\]

we will draw inspiration from interest rate models

- **N.B.:** we assume \( \exists \lambda \), a (risk-neutral) default intensity
Introduction

- there are two main classes of interest rate models
  - short rate
  - forward rate
- short rate models can be tractable
- or at least amenable to efficient numerical techniques
  - tree methods
  - finite difference methods
- forward rate models have more flexible dynamics
- but are generally difficult to compute
  - simulation methods

Short rate models

short rate models start with a process description of the short-term interest rate evolving under the risk-neutral measure

- tractable models trade flexibility for simplicity & intuition
- each has certain flaws
  - Gaussian: potentially negative probabilities
  - Affine: strictly positive rate-spread correlation
- ...but they can be considered to be locally valid
- numerical models are much more flexible
- short-rate models are inherently Markovian, so efficient to evaluate
**Gaussian model**

simplest stochastic intensity model

- the Gaussian model in §7.1 is based on the Vasicek model for interest rates
- the key dynamical assumption is that innovations in rates are normal
- Lemma 7.1 is the main analytical workhorse

\[
dx(t) = (\kappa(t) - \alpha \cdot x(t)) \ dt + \sigma(t) \ dW(t)
\]

\[
\implies \mathbb{E} \left[ e^{-\int_t^T x(s) \ ds} \mid \mathcal{F}_t \right] = e^{A(t,T) - B(t,T) \cdot x(t)}
\]

where \( A \) and \( B \) are defined in (7.10) and (7.9)

- Proposition 7.2 uses these to derive solutions for the building blocks where \( r(t) \) and \( \lambda(t) \) are combinations of two Gaussian processes

**Multifactor gaussian model**

general framework for correlation

- the factors are latent in the multifactor gaussian model
- instead we start with the risk-neutral process definitions of the zerobond values

\[
\frac{dB(t, T)}{B(t, T)} = r(t) \ dt + \bar{a}(t, T) \ d\bar{W}(t)
\]

\[
\frac{dB(t, T)}{B(t-, T)} = (r(t) + \lambda(t)) \ dt + \bar{a}(t, T) \ d\bar{W}(t) - dN(t)
\]

where the vector form of the volatility allows for a completely general description of the correlation of innovations between the curves and amongst points on the curves over time
Multifactor gaussian model

general framework for correlation

- in this setting, where $B$ and $\bar{B}$ are taken as given, the remaining building block security is

$$
e(t, T) = \bar{B}(t, T) \cdot 
\left[h(t, T) - \int_t^T \bar{a}(s, T)' \frac{\partial}{\partial T} (\bar{a}(s, T) - \bar{a}(s, T)) \ ds\right]
$$

where $h = \bar{f} - f$ is the forward hazard rate
- the last term reflects the correlation between $r$ and $\lambda$
- N.B.: I believe there are typos in (7.23) and (7.24)

Cox-Ingersoll-Ross model

- the affine model in §7.2-3 is based on the Cox-Ingersoll-Ross model for interest rates
- the key dynamical assumption is that innovations in rates are ANC: affine combinations of non-central chi-squared random variables
- say there are $n$ independent factors $x_i$, where
  $$
dx_i(t) = (\alpha_i - \beta_i \cdot x_i(t)) \ dt + \sigma_i \cdot \sqrt{x_i(t)} \ dW_i(t)
$$
  with positive parameters and $\alpha_i > \sigma_i^2 / 2$ to insure that $x_i(t) > 0$ a.s.
- analogously to the gaussian model, the main analytical workhorse for this model is
  $$
E \left[ e^{-\int_t^T \sum_i c_i \cdot x_i(s) \ ds} \bigg| \mathcal{F}_t \right] = 
\left. \sum_i \log H_1(T-t, c_i) - H_2(T-t, c_i) \cdot c_i \cdot x_i(t) \right|
$$
where $H_1$ and $H_2$ are defined in (7.29) and (7.30)
Cox-Ingersoll-Ross model

define the short rate and default intensity by

\[
\begin{align*}
    r(t) &= \sum_{i=1}^{n} w_i \cdot x_i(t) \\
    \lambda(t) &= \sum_{i=1}^{n} \bar{w}_i \cdot x_i(t)
\end{align*}
\]

with \( w_i \) and \( \bar{w}_i \) non-negative

- this guarantees that rates are non-negative
- ...but also limits the model to positive correlations
- there are a total of \( 5n + 2 \) parameters to fit to calibrate the model

with the workhorse and the result of Proposition 7.8, we can write down the values of the building block securities

\[
\begin{align*}
    B(t, T) &= e^{\sum_{i=1}^{n} \log H_{1i}(T-t, w_i) - H_{2i}(T-t, w_i) \cdot w_i \cdot x_i(t)} \\
    \bar{B}(t, T) &= e^{\sum_{i=1}^{n} \log H_{1i}(T-t, w_i + \bar{w}_i) - H_{2i}(T-t, w_i + \bar{w}_i) \cdot (w_i + \bar{w}_i) \cdot x_i(t)}
\end{align*}
\]

and

\[
\begin{align*}
    e(t, T) &= \bar{B}(t, T) \\
    \sum_{i=1}^{N} \bar{w}_i \cdot (w_i + \bar{w}_i) \cdot \left( \alpha_i + x_i(t) \cdot \frac{\partial}{\partial T} \right) H_{2i}(T-t, w_i + \bar{w}_i)
\end{align*}
\]
Tree model

the tree model in §7.4 is based on the Hull-White trinomial model for interest rates
  ▶ which in turn is based on the gaussian model
▶ there are a total of ten successors to each node
  ▶ although most of these are successors are shared—the tree is re-combining
▶ in order to prevent rates from going negative, the excursion is artificially limited by having the tree fold back on itself
▶ the risk-neutral branching probabilities are calibrated according to
  ▶ (7.78-86) to fit the moments of the dynamics
  ▶ tables 7.1-3 to incorporate the correlation
▶ note that tree models are essentially explicit schemes that implement the PDE in the next section

Tree model

Hull-White gaussian model

in the Hull-White short rate model, the mean reversion level is allowed to vary deterministically in order to facilitate calibration

\[ dr(t) = [k(t) - a \cdot r(t)] \, dt + \sigma \, dW(t) \]

the solution to this is

\[ r(t) = r^*(t) + \alpha(t) \]

where the auxiliary process, \( r^* \), is defined by \( r^*(0) = 0 \) and

\[ dr^* = -a \cdot r^* \, dt + \sigma \, dW \]

and the deterministic offset

\[ \alpha(t) = r(0) \cdot e^{-a \cdot t} + \int_0^t e^{-a \cdot (t-s)} \cdot k(s) \, ds \]
Tree model

Hull-White trinomial tree model

- the model has six parameters: $a, \sigma, \bar{a}, \bar{\sigma}, \rho$, and $\pi$
- once these are specified,
  - trees in $r^*$ and $\lambda^*$ can be constructed; and
  - $\alpha(\{0 : T\})$ and $\bar{\alpha}(\{0 : T\})$ can be fit to $B(0, \{0 : T\})$ and $\bar{B}(0, \{0 : T\})$ by forward induction, starting with the risk-free curve
- the risk-neutral default branching probability is $p = 1 - e^{-\lambda^* \Delta t}$ from each non-default node
  - the recovery in the default node is $\pi$
  - should it be needed, the local expectation of the default time is $\tau^e = t + \frac{1}{\lambda} \cdot \left( 1 - e^{\frac{\lambda^* \Delta t}{e^{\lambda^* \Delta t} - 1}} \right)$
- once calibrated, the tree can be used to price defaultable securities by backwards induction
- early exercise can be modeled by evaluating the early exercise option at each step

Partial differential equation

- to derive the partial differential equation for a defaultable security in the intensity setting, start by specifying the stochastic processes for the rates

$$
    dr = \mu_r \, dt + \sigma_r \, dW_1 \\
    d\lambda = \mu_\lambda \, dt + \sigma_\lambda \cdot \left( \rho \, dW_1 + \sqrt{1 - \rho^2} \, dW_2 \right)
$$

- also, let the compensator measure for the marked point process be

$$
    \nu(dt, d\pi) = \lambda(t) dt \, K(d\pi)
$$

- and let the cashflow densities be

$$
    \tilde{f}(t, r, \lambda) \quad \text{prior to default} \\
    g(t, r, \lambda, \pi) \quad \text{at default}
$$
define the value of the security to be

\[ V(t) = v(t, r(t), \lambda(t)) \quad \text{for} \quad t < \tau \]

applying Itô’s Lemma and the fundamental pricing rule, we get that \( v \) must satisfy

\[ \partial_t v + \mathcal{L}v - (r + \lambda) \cdot v = -g^e \cdot \lambda - \tilde{f} \]

where the diffusion’s linear operator is

\[ \mathcal{L} = \mu_r \cdot \partial_r + \frac{1}{2} \cdot \sigma^2_r \cdot \partial_{rr} + \mu_\lambda \cdot \partial_\lambda + \frac{1}{2} \cdot \sigma^2_\lambda \cdot \partial_{\lambda\lambda} + \rho \cdot \sigma_r \cdot \sigma_\lambda \cdot \partial_{r\lambda} \]

and the locally (risk-neutral) expected default payoff is

\[ g^e(t, r, \lambda) = \int_0^1 g(t, r, \lambda, \pi) K(d\pi) \]

In these terms, a valuation formula \( v \) for any defaultable claim can be found given:

1. a final condition defined in terms of the no-default payoff *(may not be so simple)*

\[ v(T, r, \lambda) = F(r, \lambda) \]

2. boundary conditions on \( v \) at the extremes of \( r \) and \( \lambda \)

the problem is comparable to that for a default-free interest-sensitive claim

in particular, adding a stochastic intensity and recovery effectively transforms:

- discount rate \( r \rightarrow r + \lambda \)
- dividend rate \( \tilde{f} \rightarrow \tilde{f} + g^e \)
Forward rates framework

- the whole curve model in §7.6 is based on Heath-Jarrow-Morton
  - all short rate models are consistent with HJM
  - HJM with deterministic volatility is equivalent to the gaussian model
- since the dynamics are essentially unlimited, need to specify the no-arbitrage drift restriction
- model is non-Markovian, so valuation is in a simulation setting
- an implementation would proceed in the following steps
  1. specify initial risk-free term structure \( f(0, \{0:T\}) \)
  2. ...zero-recovery spread term structure \( h(0, \{0:T\}) \)
  3. ...forward rate volatilities and correlations
  4. calculate drifts
  5. simulate paths in \( \beta, B, \bar{B}, e, I, \) and \( \pi \)
  6. evaluate pathwise security values
  7. average

Heath-Jarrow-Morton

we start by taking as given,

1. the rates and spreads for \( T \geq 0 \)
   \[ f(0, T) \text{ and } h(0, T) \]
2. and the \( i = 1, \ldots, n \) co-volatilities for \( t \geq 0 \) and \( T \geq t \)
   \[ \sigma_i(t, T) \text{ and } \sigma^h_i(t, T) \quad \forall t < \tau \]

- and define the risk-neutral processes for \( 0 \leq t \leq T \),
  \[ df(t, T) = \alpha(t, T) \ dt + \sum_{i=1}^{n} \sigma_i(t, T) \ dW_i \]
  \[ dh(t, T) = \alpha^h(t, T) \ dt + \sum_{i=1}^{n} \sigma^h_i(t, T) \ dW_i \quad \forall t < \tau \]
Heath-Jarrow-Morton

no-arbitrage imposes the drift restrictions

\[ \alpha(t, T) = \sum_{i=1}^{n} \sigma_i(t, T) \cdot \int_{t}^{T} \sigma_i(t', T') \, dT' \]

and

\[ \alpha^h(t, T) = \sum_{i=1}^{n} \left[ \sigma_i^h(t, T) \cdot \int_{t}^{T} \sigma_i(t', T') \, dT' \right. \]

\[ \left. + \left( \sigma_i(t, T) + \sigma_i^h(t, T) \right) \cdot \int_{t}^{T} \sigma_i^h(t', T') \, dT' \right] \]

also, Proposition 7.11 verifies that

\[ h(t, t) = \lambda(t) \]

which can be used to model the default indicator process

Monte Carlo

to value a defaultable security under an intensity model, we need to evaluate the pathwise integral in (7.143),

\[ \mathbb{E}^Q \left[ \beta(0, T) \cdot I(T) \cdot F(T) | \mathcal{F}_0 \right] \]

\[ + \mathbb{E}^Q \left[ \beta(0, \tau) \cdot N(T) \cdot g(\tau, \pi) | \mathcal{F}_0 \right] \]

\[ \mathbb{E}^Q \left[ \int_0^{T} \beta(0, t) \cdot I(t) \cdot \tilde{f}(t) \, dt | \mathcal{F}_0 \right] \]

where

- \( F \) is the final payoff at \( T \) if \( T < \tau \)
  - may depend on \( f(T, t) \) and \( h(T, t) \) for \( T \leq t < \tilde{T} \)
- \( g(t, \pi) \) is the payoff at default for recovery \( \pi \) if \( t = \tau \)
- \( \tilde{f}(t) \) is the dividend rate for \( t < \tau \land T \)

and \( \beta, \tau, N, I, \) and \( \pi \) all depend on the path \( \omega \)
Monte Carlo

scheme for rates

the HJM framework is well-suited for an Euler scheme to simulate pathwise forward rates

- time from \( t = 0 \) to \( \bar{T} \geq T \) is discretized
- forward rates are updated according to

\[
\begin{align*}
    f(t_m, t_l) &= f(t_{m-1}, t_l) + \alpha f(t_m, t_l) \cdot (t_m - t_{m-1}) \\
        &+ \sum_{i=1}^{n} \sigma_i f(t_{m}, t_{l}) \cdot \epsilon_{i,m} \cdot \sqrt{t_m - t_{m-1}}
    \\
    h(t_m, t_l) &= h(t_{m-1}, t_l) + \alpha h(t_m, t_l) \cdot (t_m - t_{m-1}) \\
        &+ \sum_{i=1}^{n} \sigma_i h(t_{m}, t_{l}) \cdot \epsilon_{i,m} \cdot \sqrt{t_m - t_{m-1}}
\end{align*}
\]

for \( l \geq m \) and \( \epsilon_{i,m} \) standard i.i.d. variates

Monte Carlo

schemes for default

by way of illustration, the book describes three schemes of increasing efficiency for simulating default

- fixed time grid
  - draw a uniform variate at each step in each path
  - default immediately if
    \[-\log U_m < h(t_m, t_m) \cdot (t_m - t_{m-1})\]
  - direct simulation of default time
    - draw a single uniform variate for each path
    - set \( \tau \) to be the lowest \( t_M \) with
      \[-\log U < \sum_{m=1}^{M} h(t_m, t_m) \cdot (t_m - t_{m-1})\]
  - simulation with branching to default
    - do not simulate default at all; apply iterated expectations instead

the author claims the latter method converges much quicker
Monte Carlo simulation with branching to default

analogously to Figure 3.2, calculate the (risk-neutral) expected defaultable value at each step in the tree for each path,

- simulate all rates $f$ and $h$ out to $\bar{T}$
- set $\beta = \gamma = 1$ and $v = 0$ at $t = 0$
- value the security along the path from $t = 0$ to $T$
  1. update the discount factor: $\beta \leftarrow \beta \cdot e^{-f(t,t) \cdot \Delta t}$
  2. calculate the default probability: $p = 1 - e^{-h(t,t) \cdot \Delta t}$
  3. update the survival probability: $\gamma \leftarrow \gamma \cdot (1 - p)$
  4. accumulate the value:
     $v \leftarrow v + \beta \cdot \gamma \cdot \left( \bar{f}(t) \cdot \Delta t \cdot (1 - p) + g^e(t) \cdot p \right)$
     - where $g^e(t) = E^Q [g(t,\pi) | F_t]$
     - N.B.: include the payoff $F(T)$ in the last step

...then average the values, $v$, over the paths

Conclusion

- questions?
- remaining chapters
  - ch. 5 Cox process Chris Bemis
  - ch. 6 recovery models
  - ch. 8 transition models John Baxter
  - ch. 9 structural model Bill Barr
  - ch. 10 correlation models Carlos Tolmasky
- papers
  - Duffie-Lando (2001), Term structures of credit spreads with incomplete accounting information
  - Andersen-Sidenius-Basu (2003), All your hedges in one basket
  - Carr-Flesaker (2006), Robust replication of default contingent claims
  - Errais-Giesecke-Goldberg (2007), Pricing credit from the top down with affine point processes
seven Wednesdays left before Fall term starts on September 4 (Labour Day is September 3)

- July 18
- July 25
- August 1
- August 8
- August 15
- August 22
- August 29 (State Fair!)