The challenge of estimating the market price of risk

John A. Dodson∗

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Consider a non-dividend-paying1 asset, whose value $S(t)$ is governed by geometric Brownian motion,

$$\frac{dS}{S} = (r(t) + \lambda \cdot \sigma) \; dt + \sigma \; dW$$

where the volatility, $\sigma$, and the market price of risk, $\lambda$, are assumed to be constants, the risk-free interest rate $r(t)$ is an adapted process, and $W$ is a standard Brownian motion.

We are justified in calling $\lambda$ the market price of risk because it parameterizes the Radon-Nikodym derivative for the change to the risk-neutral measure $Q$,

$$\frac{dQ}{dP} = e^{-\lambda \cdot W - \frac{\lambda^2}{2} \cdot t}$$

under which

$$\tilde{W} = W + \lambda \cdot t$$

is a martingale.

We are interested in estimating $\lambda$. We sample $S(t)$, along with a bank account $B(t)$ for which

$$dB = r(t) \cdot B \; dt$$

at (possibly irregular) instances, $\{t_0, t_1, \ldots, t_N = t_0 + T\}$, leading to a sample of size $N$ characterizing the continuous path over a period of $T$ years. Since increments of a Brownian motion are independent normal random variables, the $N$ quantities defined by

$$\{x_i = \log \left( \frac{S(t_i)}{S(t_{i-1})} \right) - \log \left( \frac{B(t_i)}{B(t_{i-1})} \right) \; \forall \; i = 1, \ldots, N\}$$

are independent with a population density given by

$$X_i \sim N \left( (\lambda \cdot \sigma - \frac{1}{2} \sigma^2) \cdot (t_i - t_{i-1}), \sigma^2 \cdot (t_i - t_{i-1}) \right)$$

We can therefore form the log-likelihood function for the joint estimation of the parameters $\lambda$ and $\sigma^2$,

$$L(\lambda, \sigma^2) = -\frac{1}{2} \cdot \sum_{i=1}^{N} \log \left( 2\pi \cdot \sigma^2 \cdot (t_i - t_{i-1}) \right) + \frac{\left( x_i - (\lambda \cdot \sigma - \frac{1}{2} \sigma^2) \cdot (t_i - t_{i-1}) \right)^2}{\sigma^2 \cdot (t_i - t_{i-1})}$$

∗The Options Clearing Corp. quantitative risk management
1or an asset with dividends fully re-invested without tax implication
Estimating the historical volatility is generally not a challenge. Accurately estimating the market price of risk is more difficult as we shall see. Start by considering the Fisher information for this sample,

\[ I(\lambda, \sigma^2) = -E \left[ \frac{\partial^2 \log f(X_1, \ldots, X_N | \lambda, \sigma^2)}{\partial (\lambda, \sigma^2)^2} \right] = \left( \frac{T}{2 \sigma^2} \cdot \frac{\lambda - \sigma}{2 \sigma^2} \cdot \frac{T}{N} + \left( \frac{\lambda - \sigma}{2 \sigma^2} \right)^2 \cdot T \right) \]

Notice that the particulars of the sampling protocol are condensed down to just two quantities here: the total length of time in the sample period \( T \) and the total number of observations \( N \). This has important consequences. The Cramér-Rao result says that the (co)variance of the estimate for any unbiased estimator of the parameters is bounded below by the inverse of the Fisher information,

\[ I^{-1}(\lambda, \sigma^2) = \left( \frac{1}{T} + \frac{(\lambda - \sigma)^2}{2 \sigma^4} \cdot \frac{2 \sigma^4}{N} - \frac{\lambda - \sigma}{2 \sigma^2} \cdot \frac{2 \sigma^4}{N} \right) \]

In particular, the standard error for any unbiased estimator \( \hat{\sigma}^2 \) for \( \sigma^2 \) is bounded below by

\[ \text{SE} \left( \hat{\sigma}^2 \right) \geq \sqrt{\frac{2}{N}} \cdot \sigma^2 \]

which can be made arbitrarily small by sampling more frequently\(^2\); while the standard error for any unbiased estimator \( \hat{\lambda} \) for the market price of risk is bounded below by

\[ \text{SE} \left( \hat{\lambda} \right) \geq \sqrt{\frac{1}{T} + \frac{(\lambda - \sigma)^2}{2 \cdot N}} \geq \frac{1}{\sqrt{T}} \]

regardless of the sample size.

There is an economic rationale for \( \lambda > 0 \), so a reasonable benchmark for an estimator for the market price of risk is that the standard error of the estimate be less than half of the true value. For \( \lambda \approx 1 \) this would require \( T > 4 \), at least four years of consistent data. For lower true values, even more history is required.

For this reason, practitioners faced with the challenge of estimating the market price of risk from historical data usually resort to estimators that introduce some form of bias, such as James-Stein shrinkage estimators or Bayesian estimators.

\(^2\)This is not to say that any particular estimator is efficient, only that the Cramér-Rao lower bound on the standard error can be made arbitrarily small.