Implied Option Delta

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The Black solution for an European-style option on a forward is

\[ f = \omega d \left( F \Phi (\omega z) - K \Phi (\omega z - \omega \sigma \sqrt{\tau}) \right) \]

with \[ z = \frac{\log F/K}{\sigma \sqrt{\tau}} + \frac{1}{2} \sigma \sqrt{\tau} \] (1)

Where the option term is \( \tau \) years, the discount is \( d \), the strike price is \( K \), the forward price is \( F \), the implied volatility (rate) is \( \sigma \), \( \Phi(\cdot) \) is the normal distribution, and

\[ \omega = \begin{cases} +1 & \text{call} \\ -1 & \text{put} \end{cases} \]

We are interested in eliminating the implied volatility with the option delta \( \Delta \). In this case, let’s work with the delta relative to the discounted forward:

\[ \Delta \equiv \frac{\partial f}{\partial (dF)} = \omega \Phi (\omega z) \]

[If you wish to verify this, recall the trick \( F \Phi'(z) = K \Phi'(z - \sigma \sqrt{\tau}) \).]

Toward our goal of eliminating \( \sigma \sqrt{\tau} \), note that

\[ \omega \Phi^{-1} (\Delta) = z \]

where we have used the fact that \( \omega \Delta = |\Delta| \). Hence, using the definition of \( z \), we can set-up the quadratic equation,

\[ \frac{1}{2} \omega^2 \sigma^2 \tau - \Phi^{-1} (|\Delta|) \omega \sigma \sqrt{\tau} + \log F/K = 0 \]

which we can solve to get

\[ \omega \sigma \sqrt{\tau} = \omega z \pm \sqrt{\Phi^{-1} (|\Delta|)^2 - 2 \log F/K} \]

[We only care about one solution above, but which one does not matter below.]

We can now re-write (1) in these terms:

\[ f = dF \Delta - dK \omega \Phi \left( \pm \sqrt{\Phi^{-1} (|\Delta|)^2 - 2 \log F/K} \right) \]

and from this we get

\[ \Phi^{-1} (|\Delta|)^2 - \Phi^{-1} \left( \frac{F}{K} |\Delta| - \frac{f}{\omega dK} \right)^2 = 2 \log \frac{F}{K} \] (2)

which we can use to solve (numerically) for \( \Delta = \omega |\Delta| \) without first solving for \( \sigma \).
Numerical Estimation

So we are interested in roots of an equation like

\[ h(x; A, B) \triangleq \Phi^{-1}(x)^2 - \Phi^{-1}(Ax - B)^2 - 2 \log A \]

for

\[ A > 0 \quad \text{and} \quad -1 < B < A \]  \hspace{1cm} (3)

Newton’s method

Newton’s method is effective here. Recall that for \( h(x^*) = 0 \), then for \( x \) near \( x^* \),

\[ \frac{h(x) - h(x^*)}{x - x^*} \approx h'(x) \]

so

\[ x^* \approx x - \frac{h(x)}{h'(x)} \]

if \( h'(x) \neq 0 \).

If \( x_0 \) is “near enough”, \( x_{n+1} \triangleq x_n - h(x_n)/h'(x_n) \) is in the domain of \( h(\cdot) \) and \( h'(x_n) \neq 0 \) for \( n = 0, 1, \ldots \), then

\[ h \left( \lim_{n \to \infty} x_n \right) = 0 \]

In this case,

\[ h'(x) = \frac{2\Phi^{-1}(x)}{\Phi'\left(\Phi^{-1}(x)\right)} - \frac{2A\Phi^{-1}(Ax - B)}{\Phi'\left(\Phi^{-1}(Ax - B)\right)} \]  \hspace{1cm} (4)

Since the range of \( \Phi(\cdot) \) is \((0, 1)\), the domain of \( \Phi^{-1}(\cdot)^2 \) is also \((0, 1)\). Therefore, the domain of \( h(x) \) is \( \max(0, B/A) < x < \min(1, B/A + 1/A) \), which is guaranteed to be non-empty by (3). Also, since there is no solution to \( x = Ax - B \) in the domain as long as

\[ B > \max(0, A - 1) \quad \text{or} \quad B < \min(0, A - 1) \]  \hspace{1cm} (5)

and since \( \Phi^{-1}(\cdot)^2 \) is convex up, we can conclude that \( h'(\cdot) \neq 0 \). Therefore at least we know that Newton’s method does not fail here (although we may need to adjust \( x_{n+1} \) to keep it in the domain).

I cannot prove that it always converges for this problem (although in practice it seems to), but a reasonable starting value seems to be

\[ x_0 \triangleq \frac{\min(1, B/A + 1/A) + \max(0, B/A)}{2} \]

American-style Options

The constraints (5) subject to (3) in terms of (2) amount to

\[ f > \begin{cases} 
\max(0, F - K) & \text{call} \\
\max(0, K - F) & \text{put}
\end{cases} \]  \hspace{1cm} (6)

for \( F, K > 0 \), which is reasonable for European-style options.
Figure 1: Parameter region with a solution to (8)

For options with early-exercise rights, this is no longer reasonable. We need to modify it to

\[
f' \geq \begin{cases} 
\max(0, S - K) & \text{call} \\
\max(0, K - S) & \text{put}
\end{cases}
\]

(7)

where \(f'\) is the premium and \(S\) is the spot price. This represents the fact that the effective term may be less than \(\tau\), and may in fact be zero.

This suggests a modification to (2):

\[
\Phi^{-1} (|\Delta|)^2 - \Phi^{-1} \left( \frac{S}{K} |\Delta| - \frac{f'}{\omega K} \right)^2 = 2 \log \frac{S}{K}
\]

(8)

where we handle the equality in (7) by exception (\(f' = 0 \Rightarrow \Delta = 0\), \(\omega f' = S - K \Rightarrow \Delta = \omega\)).