Attilio Meucci
Lehman Brothers, Inc., New York

personal website: symmys.com

Issues in Quantitative Portfolio Management: Handling Estimation Risk
AGENDA

Estimation vs. Modeling

Classical Optimization and Estimation Risk

Black-Litterman Optimization

Robust Optimization

Bayesian Optimization

Robust Bayesian Optimization

References
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References
ESTIMATION vs. MODELING – general conceptual framework

Estimation = Invariance (i.i.d.) Detection

Projection

Modeling & Optimization

time series analysis

investment horizon

P&L

investment decision

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ESTIMATION vs. MODELING – fixed-income PCA trading recipe

1. consider N series of T observations of homogeneous forward rates
   \[ X \quad \text{(TxN panel)} \]

2. define \( S \equiv X'X \quad \text{(NxN positive definite matrix)} \)

3. run PCA \( S \equiv \mathbf{E}\Lambda\mathbf{E}' \quad \text{(eigenvectors-eigenvalues-eigenvectors)} \)
ESTIMATION vs. MODELING – fixed-income PCA trading recipe

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   \[ X \] (TxN panel)

2. define \[ S \equiv X'X \] (NxN positive definite matrix)

3. run PCA \[ S \equiv E \Lambda E' \] (eigenvectors-eigenvalues-eigenvectors)

4. analyze the series \( y \equiv X e^{(N)} \) of the last factor
   • z-score: structural bands
   • “juice”: b.p. from mean
   • roll-down/slide-adjusted prospective Sharpe ratio
   • reversion timeframe
   • market events (e.g. Fed, Thursday “numbers”,…)
   • relation with other series (e.g. oil prices)

5. convert basis points to PnL/risk exposure by dv01

   variations: transform series, include mean, support series (PCA-regression),…

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5. convert basis points to PnL/risk exposure by dv01

• estimation (backward-looking) and projection/modeling (forward-looking) overlap
  • non-linearities not accounted for
1. consider N series of T observations of fund prices $P$ (TxN panel)

2. consider the compounded returns $C_{t,n} \equiv \ln(P_{t,n}) - \ln(P_{t-1,n})$

3. estimate covariance (e.g. the sample non-central) \[ \hat{\Sigma} \equiv \frac{1}{T} \sum_{t=1}^{T} C_t C_t^\prime \]

4. define the expected values (e.g. risk-premium) \[ \hat{\mu} \equiv \gamma \text{diag}(\hat{\Sigma}) \]
1. consider N series of T observations of fund prices $P$ (TxN panel)

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5. solve mean-variance: $w^{(i)} \equiv \arg\max \left\{ w' \hat{\mu} \right\}$

6. choose the most suitable combination among $w^{(i)}$ according to preferences

**Estimation vs. Modeling – fund of funds flawed management recipe**

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1. consider N series of T observations of fund prices $P$ (TxN panel)

2. consider the compounded returns $C_{t,n} \equiv \ln(P_{t,n}) - \ln(P_{t-1,n})$

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5. solve mean-variance: $w^{(i)} \equiv \arg\max_{w \in \mathcal{C}} \left\{ w' \hat{\mu} \right\}$

6. choose the most suitable combination among $w^{(i)}$ according to preferences

- estimation (backward-looking) and modeling (forward-looking) overlap
- projection (investment horizon) not accounted for
- non-linearities of compounded returns not accounted for
Estimation: compounded returns

\[ C_t \equiv \ln(P_t) - \ln(P_{t-\tau}) \]

compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion.
ESTIMATION vs. MODELING – fund of funds **consistent** management recipe

- **Estimation**: compounded returns
  \[ C_t^{\tau} = \ln(P_t) - \ln(P_{t-\tau}) \]

  compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion

- **Projection to investment horizon**
  \[ C_t^{\tau} = C_{t-j\tau}^{\tau} + C_{t-(j-1)\tau}^{\tau} + \cdots + C_t^{\tau} \]

  compounded returns can be easily projected to the investment horizon because they are additive ("accordion" expansion)
ESTIMATION vs. MODELING – fund of funds **consistent** management recipe

- **Estimation:** compounded returns \( C_{t}^\tau \equiv \ln(P_t) - \ln(P_{t-\tau}) \)

  compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion

- **Projection to investment horizon** \( C_t^\tau = C_{t-J}^\tau + C_{t-(J-1)}^\tau + \cdots + C_t^\tau \)

  compounded returns can be easily projected to the investment horizon because they are additive (“accordion” expansion)

- **Modeling:** linear returns \( L_t^\tau \equiv P_t / P_{t-\tau} - 1 \)

  linear returns are related to portfolio quantities (P&L): \( L_{\Pi} = w' L \)

  securities’ returns

  securities’ relative weights

  portfolio return

  compounded returns are NOT related to portfolio quantities (P&L): \( C_{\Pi} \neq w'C \)
ESTIMATION vs. MODELING – fund of funds consistent management recipe

- **Estimation:** compounded returns
  \[ C_{t}^{\hat{r}} \equiv \ln(P_t) - \ln(P_{t-\hat{r}}) \]

  sample/risk-premium:
  \[ \hat{\Sigma}^{\hat{r}} \equiv \frac{1}{T} \sum_{t=1}^{T} C_{t}^{\hat{r}} C_{t}^{\hat{r}}' \]
  \[ \hat{\mu} \equiv \gamma \text{ diag} \left( \hat{\Sigma}^{\hat{r}} \right) \]

- **Projection to investment horizon**
  \[ C_{t}^{\tau} \equiv C_{t-J\hat{r}}^{\hat{r}} + C_{t-(J-1)\hat{r}}^{\hat{r}} + \cdots + C_{t}^{\hat{r}} \]

- **Modeling:** linear returns
  \[ L_{t}^{\tau} \equiv \frac{P_t}{P_{t-\tau}} - 1 \]

  Black-Scholes assumption: (log-normal)
  \[ m_n \equiv E \left\{ L_{t,n}^{\tau} \right\} = e^{\hat{r}\left(\frac{\tau}{2} \mu_n + \frac{1}{2} \Sigma_n^{\tau} \right)} \]
  \[ S_{nm} \equiv \text{Cov} \left\{ L_{t,n}^{\tau}, L_{t,m}^{\tau} \right\} = e^{\hat{r}\left(\frac{\mu_n^{\tau} + 1}{2} \Sigma_m^{\tau} + \mu_m^{\tau} + 1 \Sigma_m^{\tau} \right) \left( e^{\hat{r} \Sigma_m^{\tau}} - 1 \right)} \]

  the mean - variance optimization can be fed with the appropriate inputs
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CLASSICAL OPTIMIZATION – mean-variance in theory …

\[ w^{(i)} \equiv \arg\max \left\{ w' m \right\} \]

subject to

\[ w \in C \]
\[ w' S w \leq \nu^{(i)} \]

\[ w \]: relative portfolio weights

\[ C \]: set of investment constraints, e.g. \[ w' I = 1, \ w \geq 0 \]

\[ \nu^{(i)} \]: significant grid of target variances

\[ m \equiv E \left\{ L_{t+\tau}^\tau \right\} \]

\[ S \equiv \text{Cov} \left\{ L_{t+\tau}^\tau \right\} \]

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CLASSICAL OPTIMIZATION – … mean-variance in practice

\[ w^{(i)} \equiv \arg \max_{w} \{ w' m \} \]

subject to
\[ \begin{align*}
    w & \in C \\
    w' S w & \leq \nu^{(i)}
\end{align*} \]

\[ w^{(i)} \equiv \arg \max_{w} \{ w' \hat{m} \} \]

subject to
\[ \begin{align*}
    w & \in C \\
    w' \hat{S} w & \leq \nu^{(i)}
\end{align*} \]

\( w \): relative portfolio weights

\( C \): set of investment constraints, e.g. \( w' I = 1, \quad w \geq 0 \)

\( \nu^{(i)} \): significant grid of target variances

\( m \equiv E \{ L_{t+\tau}^\tau \} \)

\( S \equiv \text{Cov} \{ L_{t+\tau}^\tau \} \)

\( \hat{m} \): estimate of \( m \)

\[ \hat{m} \equiv \frac{1}{T} \sum_{t=1}^{T} l_t^\tau \]

\( \hat{S} \): estimate of \( S \)

\[ \hat{S} \equiv \frac{1}{T} \sum_{t=1}^{T} (l_t^\tau - \hat{m})(l_t^\tau - \hat{m})' \]
CLASSICAL OPTIMIZATION – estimation risk

The true optimal allocation is determined by a set of parameters that are estimated with some error:

\[ \hat{\theta} \equiv (\hat{m}, \hat{S}) \neq \theta \equiv (m, S) \]
CLASSICAL OPTIMIZATION – estimation risk

The true optimal allocation is determined by a set of parameters that are estimated with some error:

$$\hat{\theta} \equiv (\hat{m}, \hat{S}) \neq \theta \equiv (m, S)$$

- The classical “optimal” allocation based on point estimates $\hat{\theta} \equiv (\hat{m}, \hat{S})$ is sub-optimal
- More importantly, the sub-optimality due to estimation error is large (Jobson & Korkie (1980); Best & Grauer (1991); Chopra & Ziemba (1993))
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BLACK-LITTERMAN APPROACH – inputs: prior

linear returns \( L \sim N(\mu, \Sigma) \)

unconstrained Markowitz mean-variance optimization

\[
\begin{align*}
\mathbf{w}_\varsigma & \equiv \arg\max \left\{ \mathbf{w}' \mu - \frac{1}{2\varsigma} \mathbf{w}' \Sigma \mathbf{w} \right\} \\
\mathbf{w}_\varsigma & \equiv \varsigma \Sigma^{-1} \mu
\end{align*}
\]

market-implied equilibrium prior
expected returns

average risk propensity \( \sim 0.4 \)

well-diversified portfolio
(equilibrium - benchmark)

exponentially smoothed estimate of covariance

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BLACK-LITTERMAN APPROACH – outputs: posterior

“official” prior on linear returns \( L \sim N(\mu, \Sigma) \) ← equilibrium-based estimation

subjective views

\[
\begin{align*}
V_1 & \equiv p_1 \mu \sim N(q_1, \omega_1^2) \\
\vdots \\
V_K & \equiv p_K \mu \sim N(q_K, \omega_K^2)
\end{align*}
\]

Bayesian posterior: \( \mu_{BL} \equiv \mu + \Sigma P' (P \Sigma P' + \Omega)^{-1} (q - P \mu) \)

\( P \equiv \begin{bmatrix} p_1 \\ \vdots \\ p_K \end{bmatrix} \)

matrix of stacked “pick” row-vectors
BLACK-LITTERMAN APPROACH – outputs: portfolios

“official” prior on linear returns \( L \sim N(\mu, \Sigma) \) ← equilibrium-based estimation

subjective views

\[
\begin{align*}
V_1 &\equiv p_1\mu \sim N(q_1, \omega_1^2) \\
&\vdots \\
V_K &\equiv p_K\mu \sim N(q_K, \omega_K^2)
\end{align*}
\]

Bayesian posterior: \( \mu_{BL} \equiv \mu \) + views

Markowitz mean-variance optimization:

\[
\begin{aligned}
w^* &\equiv \arg\max_{w' \mathbb{I}_{\geq 0}, w' \mathbb{I}_{\geq 1}} \left\{ w' \mu_{BL} - \frac{w' \Sigma w}{2\zeta} \right\} \\
\text{shrinkage to equilibrium}
\end{aligned}
\]

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• The point estimate for the parameters must be replaced by an uncertainty region that includes the true, unknown parameters:

$$\hat{\Theta} \equiv (\hat{m}, \hat{S}) \mapsto \Theta$$
ROBUST OPTIMIZATION – the general framework

• The point estimate for the parameters must be replaced by an uncertainty region that includes the true, unknown parameters:
  \[ \hat{\Theta} \equiv (\hat{m}, \hat{S}) \quad \mapsto \quad \Theta \]

• The allocation optimization must be performed over all the parameters in the uncertainty region:
  \[ w(i) \equiv \arg\max_{w \in C} \{ \ldots \} \quad \mapsto \quad w(i) \equiv \arg\max_{w \in C} \{ \ldots \} \]

where \( (m, S) \equiv (\hat{m}, \hat{S}) \in \Theta \).

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ROBUST OPTIMIZATION – from the standard mean-variance ...

\[
\mathbf{w}^{(i)} \equiv \arg\max_{\mathbf{w}} \left\{ \mathbf{w}' \hat{\mathbf{m}} \right\}
\]

subject to

\[
\begin{align*}
\mathbf{w} &\in \mathcal{C} \\
\mathbf{w}' \hat{\mathbf{S}}\mathbf{w} &\leq \nu^{(i)}
\end{align*}
\]

\[\mathbf{w}\] : relative portfolio weights

\[\mathcal{C}\] : set of investment constraints, e.g. \(\mathbf{w}' \mathbf{1} = 1, \mathbf{w} \geq 0\)

\[\nu^{(i)}\] : significant grid of target variances

\[\hat{\mathbf{m}}\] : (point) estimate of \(\mathbf{m}\)

\[\hat{\mathbf{S}}\] : (point) estimate of \(\mathbf{S}\)

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ROBUST OPTIMIZATION – … to a conservative mean-variance approach

\[ w^{(i)} = \arg\max_w \left\{ w' \hat{m} \right\} \]
subject to \( w \in \mathcal{C} \)
\( w' \hat{S} w \leq \nu^{(i)} \)

\[ w^{(i)} = \arg\max_w \left\{ \min_{m \in \Theta_m} w' m \right\} \]
subject to \( w \in \mathcal{C} \)
\[ \max_{S \in \Theta_S} \left\{ w' S w \right\} \leq \nu^{(i)} \]

\( w \) : relative portfolio weights
\( \mathcal{C} \) : set of investment constraints, e.g. \( w' 1 = 1, \ w \geq 0 \)
\( \nu^{(i)} \) : significant grid of target variances
\( \hat{m} \) : (point) estimate of \( m \)
\( \Theta_m \) : uncertainty set for \( m \)
\( \hat{S} \) : (point) estimate of \( S \)
\( \Theta_S \) : uncertainty set for \( S \)
ROBUST OPTIMIZATION – uncertainty regions

Trade-off for the choice of the uncertainty regions:

- Must be as **large** as possible, in such a way that the true, unknown parameters (most likely) are captured
- Must be as **small** as possible, to avoid trivial and nonsensical results

**EXAMPLE: 2x2 COVARIANCE MATRIX**

\[
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{12} & S_{22}
\end{pmatrix}
\]
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BAYESIAN OPTIMIZATION – Bayesian estimation theory

The Bayesian approach to estimation of the generic market parameters $\theta \equiv (m, S)$ differs from the classical approach in two respects:

- it blends **historical information** from time series analysis with **experience**
- the outcome of the **estimation** process is a (posterior) **distribution**, instead of a number

![Diagram showing Bayesian estimation process](image)
in the Bayesian approach the expected values of the returns are a random variable

EXAMPLE: 2-dim EXPECTED VALUES

\[ m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \]
in the Bayesian approach the covariance matrix of the returns is a random variable

EXAMPLE: 2x2 COVARIANCE MATRIX

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S = \begin{pmatrix}
S_{11} & S_{12} \\
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\]
BAYESIAN OPTIMIZATION – from the standard mean-variance …

\[ w^{(i)} \equiv \arg\max_w \left\{ w' \hat{m} \right\} \]

subject to \( w \in \mathcal{C} \)

\[ w' \hat{S} w \leq \nu^{(i)} \]

\( w \) : relative portfolio weights

\( \mathcal{C} \) : set of investment constraints, e.g. \( w' I = 1, \ w \geq 0 \)

\( \nu^{(i)} \) : significant grid of target variances

\( \hat{m} \) : classical estimate of \( m \)

\( \hat{S} \) : classical estimate of \( S \)
BAYESIAN OPTIMIZATION – … to the classical-equivalent mean-variance

\[ w^{(i)} \equiv \arg \max_w \left\{ w' \hat{m} \right\} \]

subject to

\[ w \in \mathcal{C} \]
\[ w' \hat{S} w \leq \nu^{(i)} \]

\[ w^{(i)} \equiv \arg \max_w \left\{ w' \hat{m}_{ce} \right\} \]

subject to

\[ w \in \mathcal{C} \]
\[ w' \hat{S}_{ce} w \leq \nu^{(i)} \]

\[ w \] : relative portfolio weights

\[ \mathcal{C} \] : set of investment constraints, e.g. \( w' I = 1, \ w \geq 0 \)

\[ \nu^{(i)} \] : significant grid of target variances

\[ \hat{m} \] : classical estimate of \( m \)

\[ \hat{m}_{ce} \] : Bayesian classical-equivalent estimate for \( m \)

\[ \hat{S} \] : classical estimate of \( S \)

\[ \hat{S}_{ce} \] : Bayesian classical-equivalent estimate for \( S \)
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Robust allocations are guaranteed to perform adequately for all the markets within the given uncertainty ranges.

Bayesian allocations include the practitioner’s experience.
Robust allocations are guaranteed to perform adequately for all the markets within the given uncertainty ranges. However…

• the uncertainty regions for the market parameters are somewhat arbitrary
• the practitioner’s experience, or prior knowledge, is not considered

Bayesian allocations include the practitioner’s experience. However…

• the approach is not robust to estimation risk

Bayesian approach to parameter estimation within the robust framework
ROBUST BAYESIAN OPTIMIZATION – Bayesian ellipsoids

The Bayesian posterior distribution defines naturally a self-adjusting uncertainty region $\widehat{q}$ for the market parameters.

This region is the location-dispersion ellipsoid defined by

- a location parameter: the classical-equivalent estimator $\hat{\Theta}_{ce}$
- a dispersion parameter: the positive symmetric scatter matrix $S_\theta$
- a radius factor $q$ $\widehat{\Theta}^q : \left( \Theta - \hat{\Theta}_{ce} \right)' S_\theta^{-1} \left( \Theta - \hat{\Theta}_{ce} \right) \leq q^2$

posterior density $f_{p.o}(\Theta)$

uncertainty region $\widehat{\Theta}^q$

space of possible parameters values

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**ROBUST BAYESIAN OPTIMIZATION – Bayesian ellipsoids**

Standard choices for the classical equivalent and the scatter matrix respectively:

- global picture: expected value / covariance matrix

\[
\hat{\Theta}_{ce} \equiv \int \Theta f_{po}(\Theta) \, d\Theta
\]

\[
S_\Theta \equiv \int \left( \Theta - \hat{\Theta}_{ce} \right) \left( \Theta - \hat{\Theta}_{ce} \right)' f_{po}(\Theta) \, d\Theta
\]

- local picture: mode / modal dispersion

\[
\hat{\Theta}_{ce} \equiv \arg\max_{\Theta} \{ f_{po}(\Theta) \}
\]

\[
S_\Theta \equiv -\left( \frac{\partial \ln f_{po}(\Theta)}{\partial \Theta \partial \Theta'} \bigg|_{\hat{\Theta}_{ce}} \right)^{-1}
\]
ROBUST BAYESIAN OPTIMIZATION – from the standard mean-variance …

\[ w^{(i)} \equiv \arg\max_{w \in \mathcal{C}} \left\{ w' \hat{m} \right\} \]

subject to \( w' \hat{S} w \leq \nu^{(i)} \)

\( w \) : relative portfolio weights

\( \mathcal{C} \) : set of investment constraints, e.g. \( w' I = 1, \ w \geq 0 \)

\( \nu^{(i)} \) : significant grid of target variances

\( \hat{m} \) : (point) estimate of \( m \)

\( \hat{S} \) : (point) estimate of \( S \)
ROBUST BAYESIAN OPTIMIZATION – … to the robust mean-variance …

\[
\begin{align*}
&\argmax_{w \in \mathcal{C}} \left\{ w' \hat{m} \right\} \quad \argmax_{w \in \mathcal{C}} \left\{ \min_{m \in \Theta_m} w' m \right\} \\
&\text{subject to } w' \hat{S} w \leq \nu^{(i)} \quad \text{subject to } \max_{S \in \Theta_S} \left\{ w' S w \right\} \leq \nu^{(i)}
\end{align*}
\]

\( w \) : relative portfolio weights

\( \mathcal{C} \) : set of investment constraints, e.g. \( w' \mathbf{1} = 1, \ w \geq 0 \)

\( \nu^{(i)} \) : significant grid of target variances

\( \hat{m} \) : (point) estimate of \( m \)

\( \hat{\Theta}_m \) : uncertainty set for \( m \)

\( \hat{S} \) : (point) estimate of \( S \)

\( \hat{\Theta}_S \) : uncertainty set for \( S \)
ROBUST BAYESIAN OPTIMIZATION – … to the robust Bayesian MV

\[ w^{(i)} \equiv \arg\max_{w \in \mathcal{C}} \{ w' \hat{m} \} \]

subject to \( w' \hat{S} w \leq \nu^{(i)} \)

\[ w^{(i)} \equiv \arg\max_{w \in \mathcal{C}} \left\{ \min_{m \in \hat{\Theta}_m} w' m \right\} \]

subject to \( \max_{S \in \hat{\Theta}_S} \{ w' S w \} \leq \nu^{(i)} \)

\[ w \] : relative portfolio weights

\[ \mathcal{C} \] : set of investment constraints, e.g. \( w' I = 1, \ w \geq 0 \)

\[ \nu^{(i)} \] : significant grid of target variances

\[ \hat{m} \] : (point) estimate of \( m \)

\[ \hat{\Theta}_m \] : Bayesian ellipsoid of radius \( q \) for \( m \)

\[ \hat{S} \] : (point) estimate of \( S \)

\[ \hat{\Theta}_S \] : Bayesian ellipsoid of radius \( p \) for \( S \)
ROBUST BAYESIAN OPTIMIZATION – 3-dim. mean-variance frontier

The robust Bayesian efficient allocations \( w_{p,q}^{(i)} \) represent a **three-dimensional frontier** parametrized by:

1. **Exposure to market risk** represented by the target variance \( v^{(i)} \)

2. **Aversion to estimation risk** for the expected returns \( m \) represented by radius \( q \)

   ...indeed, a large ellipsoid \( \Theta_m^{q} \) corresponds to an investor that is very worried about poor estimates of \( m \)

3. **Aversion to estimation risk** for the returns covariance \( S \) represented by radius \( p \)

   ...indeed, a large ellipsoid \( \Theta_S^{p} \) corresponds to an investor that is very worried about poor estimates of \( S \)
We make the following assumptions:

- The market is composed of equity-like securities, for which the returns are independent and identically distributed across time.
- The estimation interval coincides with the investment horizon.
- The linear returns are normally distributed:

\[
L_{t+\tau}^\tau | m, S \sim N(m, S)
\]

We model the investor’s prior as a normal-inverse-Wishart distribution:

\[
m | S \sim N\left( m_0, \frac{S}{T_0} \right), \quad S^{-1} \sim W\left( \nu_0, \frac{S_0^{-1}}{\nu_0} \right)
\]

where

\[
(m_0, S_0) \text{: investor’s experience on } (m, S)
\]

\[
(T_0, \nu_0) \text{: investor’s confidence on } (m_0, S_0)
\]
RBO EXAMPLE – posterior distribution of market parameters

Under the above assumptions, the posterior distribution is normal-inverse-Wishart, see e.g. Aitchison and Dunsmore (1975):

\[ m \mid S \sim N \left( m_1, \frac{S}{T_1} \right), \quad S^{-1} \sim W \left( \nu_1, \frac{S^{-1}}{\nu_1} \right) \]

where

\[ \hat{m} \equiv \frac{1}{T} \sum_{t=1}^{T} l_t^\varepsilon \]

\[ T_1 \equiv T_0 + T \]

\[ \hat{m}_1 \equiv \frac{1}{T_1} \left[ T_0 m_0 + T \hat{m} \right] \]

\[ \hat{S} \equiv \frac{1}{T} \sum_{t=1}^{T} (l_t^\varepsilon - \hat{m})(l_t^\varepsilon - \hat{m})' \]

\[ \nu_1 \equiv \nu_0 + T \]

\[ \hat{S}_1 \equiv \frac{1}{\nu_1} \left[ \nu_0 S_0 + T \hat{S} + \frac{(m_0 - \hat{m})(m_0 - \hat{m})'}{1/T_0 + 1/T} \right] \]
The certainty equivalent and the scatter matrix for the posterior (Student t) marginal distribution of $m$ are computed in Meucci (2005):

$$m_{ce} = m_1, \quad S_m = \frac{1}{T_1} \frac{\nu_1}{\nu_1 - 2} S_1$$

The certainty equivalent and the scatter matrix for the posterior (inverse-Wishart) marginal distribution of $S$ are computed in Meucci (2005):

$$S_{ce} = \frac{\nu_1}{\nu_1 + N + 1} S_1, \quad S_S = \frac{2\nu_1^2}{(\nu_1 + N + 1)^3} \left( D_N' \left( S_1^{-1} \otimes S_1^{-1} \right) D_N \right)^{-1}$$

where $D_N$ is the duplication matrix (see Magnus and Neudecker, 1999) and $\otimes$ is the Kronecker product.
RBO EXAMPLE – efficient frontier

Under the above assumptions the robust Bayesian mean-variance problem:

\[
\begin{align*}
\mathbf{w}^{(i)}_{p,q} & \equiv \underset{\mathbf{w} \in \mathcal{C}}{\text{argmax}} \left\{ \underset{\mathbf{m} \in \tilde{\Theta}_m}{\text{min}} \mathbf{w}' \mathbf{m} \right\} \\
\text{subject to} \quad & \underset{\mathbf{S} \in \tilde{\Theta}_S}{\text{max}} \left\{ \mathbf{w}' \mathbf{S} \mathbf{w} \right\} \leq \nu^{(i)}
\end{align*}
\]

…simplifies as follows:

\[
\begin{align*}
\mathbf{w}^{(i)}_{p,q} & \subset \mathbf{w}(\lambda) \equiv \underset{\mathbf{w} \in \mathcal{C}}{\text{argmax}} \left\{ \mathbf{w}' \mathbf{m}_1 - \lambda \sqrt{\mathbf{w}' \mathbf{S}_1 \mathbf{w}} \right\}
\end{align*}
\]

• The **three-dimensional frontier** collapses to a **line**
• The efficient frontier is parametrized by the exposure to **overall risk**, which includes **market risk**, **estimation risk** for \( \mathbf{m} \) and **estimation risk** for \( \mathbf{S} \)

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RBO EXAMPLE – efficient frontier

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RBO EXAMPLE – robust Bayesian self-adjusting nature

- When the number of historical observations is large the uncertainty regions collapse to classical sample point estimates:

\[ w(\lambda) \equiv \arg\max_{w \in \mathcal{C}} \left\{ w' \hat{m} - \lambda \sqrt{w' \hat{S} w} \right\} \]

robust Bayesian frontier = classical sample-based frontier

- When the confidence in the prior is large the uncertainty regions collapse to the prior parameters:

\[ w(\lambda) \equiv \arg\max_{w \in \mathcal{C}} \left\{ w' m_0 - \lambda \sqrt{w' S_0 w} \right\} \]

robust Bayesian frontier = “a-priori” frontier (no information from the market)
RBO EXAMPLE – robust Bayesian self-adjusting nature

robust Bayesian frontier

portfolio weights

market & estimation risk

$T \ll T_0, \nu_0$

$T \gg T_0, \nu_0$

prior frontier

sample-based frontier
RBO EXAMPLE – robust Bayesian conservative nature (S&P 500)

Sample-based

Robust Bayesian

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RBO EXAMPLE – robust Bayesian conservative nature (S&P 500)
AGENDA

Estimation vs. Modeling

Classical Optimization and Estimation Risk

Black-Litterman Optimization

Robust Optimization

Bayesian Optimization

Robust Bayesian Optimization

References
REFERENCES

This presentation:
symmys.com > Teaching > Talks > Issues in Quantitative Portfolio Management: Handling Estimation Risk

implementation code (MATLAB):
symmys.com > Book > Downloads > MATLAB

Comprehensive discussion of
- modeling
- estimation
- location-dispersion ellipsoid
- satisfaction maximization
- quantitative portfolio-management
- risk-management
- estimation risk
- Black-Litterman allocation
- Bayesian techniques
- robust techniques
- …