Evaluating Allocations
Practitioner Course: Portfolio Optimization

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Modeling Preferences

Now that we have techniques for characterizing the market, $P_{T+\tau}$, we turn to models for investor satisfaction that will lead us into optimization. In the author’s fashion we will approach this implicitly by laying out properties that we should desire in a model for satisfaction.

Considerations

There are several risks to consider:

- How do we calibrate such a model? A survey? *Homo economicus?*
- What if the investor changes his or her mind?
- Can we assume that the investor is willing to ignore short-term results?
Investor’s Objective

Measuring Performance

Put yourself in the shoes of an investment manager faced with a potential client. The first thing you need to consider to make your case is, how will the client measure your performance?

Objective

- wealth?
- profit?
- profit relative to some benchmark portfolio?

We will represent the value of the objective at $T + \tau$ by the random variable $\Psi_\alpha$ where $\alpha$ represents the allocation that is implemented at $T$. 
Investor’s Objective

In all of the cases we will consider, we can write

$$\Psi_\alpha = \alpha' \cdot M \quad \text{with} \quad M = a + B \cdot P_{T+\tau}$$

for constants $a$ and $B$

- **wealth**: $a = 0, \ B = I$
- **profit**: $a = -p_T, \ B = I$
- **relative profit**: $a = 0, \ B = I - \frac{p_T \cdot \beta'}{\beta' \cdot p_T}$ for benchmark portfolio allocation $\beta$

Therefore, the objective is both **homogeneous** and **additive** in the allocation.

$$\Psi_{\lambda \cdot \alpha} = \lambda \cdot \Psi_\alpha \quad \Psi_{\alpha + \beta} = \Psi_\alpha + \Psi_\beta$$

**N.B.**: None of our definitions of total return has these properties.
Delta-Gamma Approximation

We can extend our characterization of the market invariants to the market vector $M = a + B \cdot g(X)$.

**Delta-Gamma Approximation**

If we can approximate the market invariants by $X \sim \mathcal{N}(\mu, \Sigma)$ and the map, $g$, by an order-two polynomial, then there is an analytic characterization of the investor’s objective.

\[
\phi_\psi(\tau) \approx \left| I - i \cdot \tau \cdot \Gamma_\alpha \cdot \Sigma \right|^{-\frac{1}{2}} \cdot e^{i \cdot \tau \cdot \left( \Theta_\alpha + \Delta'_\alpha \cdot \mu + \frac{1}{2} \cdot \mu' \cdot \Gamma_\alpha \cdot \mu \right)} \\
\cdot e^{-\frac{1}{2} \cdot (\Delta_\alpha + \Gamma_\alpha \cdot \mu)' \cdot \Sigma \cdot (I - i \cdot \tau \cdot \Gamma_\alpha \cdot \Sigma)^{-1} \cdot (\Delta_\alpha + \Gamma_\alpha \cdot \mu)} (*)
\]

where

\[
\Delta_\alpha = \alpha' \cdot B \cdot \left. \frac{\partial g}{\partial X} \right|_{0} \\
\Theta_\alpha = \alpha' \cdot (a + B \cdot g(0)) \\
\Gamma_\alpha = \alpha' \cdot B \cdot \left. \frac{\partial^2 g}{\partial X \partial X'} \right|_{0}
\]
Index of Satisfaction

The investor’s objective for a given allocation, $\Psi_\alpha$, is a scalar quantity that provides a clear ordinal ranking of outcomes at time $T + \tau$. Since it is random today at time $T$, we can characterize it, but we cannot use it to rank potential allocations. To do this, we will introduce an operator $S$, that transforms the characterization of the investor’s objective into a scalar measuring the investor’s satisfaction with a potential allocation today at time $T$.

Illustration

For example, it is not unreasonable to consider

$$S(\alpha) = \mathbb{E}\Psi_\alpha = \alpha' \cdot \mathbb{E}(M)$$

today’s expected value of the investor’s objective.

Before proceeding with this choice, let us lay out a “wish list” of properties that we would like such an operator to possess.
The goal of the satisfaction operator, $\alpha \mapsto S(\alpha)$, is to rank allocations. Let us preview some possible qualities.

- **money-equivalent** $S(\alpha)$ and $\Psi_\alpha$ are in the same units
- **estimable** $S(\alpha)$ is non-random
- **sensible** $P \{ \Psi_\alpha \geq \Psi_\beta \} = 1 \Rightarrow S(\alpha) \geq S(\beta)$
- **constant** $P \{ \Psi_b = \psi_b \} = 1 \Rightarrow S(b) = \psi_b$
- **positive homogeneous** $\lambda \geq 0 \Rightarrow S(\lambda \cdot \alpha) = \lambda \cdot S(\alpha)$
- **translation invariant** $S(\alpha + b) = S(\alpha) + \psi_b$
- **super-additive** $S(\alpha + \beta) \geq S(\alpha) + S(\beta)$
- **co-monotonic additive** $h(\cdot)$ invertible, increasing
  $\Psi_\beta = h(\Psi_\alpha) \Rightarrow S(\alpha + \beta) = S(\alpha) + S(\beta)$
- **concave** $0 \leq \lambda \leq 1 \Rightarrow$
  $S(\lambda \cdot \alpha + (1 - \lambda) \cdot \beta) \geq \lambda \cdot S(\alpha) + (1 - \lambda) \cdot S(\beta)$
- **risk averse** $E \Psi_f = 0 \Rightarrow S(f + b) \leq S(b)$
Coherent Properties

Discussion

- Constancy, translation invariance, and risk aversion all refer to the role of cash in the portfolio.
- Risk aversion says that the investor should prefer cash to any risky portfolio with the same expected outcome.
- Super-additivity and concavity both refer to the investor’s preference for diversification.
- Co-monotonic additivity means that derivatives or leverage in the portfolio provide no diversification benefit relative to underlyings (for known volatility).
- Positive homogeneity means that the satisfaction can be decomposed into the sum of marginal satisfactions.

$$S(\alpha) = \alpha' \cdot \frac{\partial S}{\partial \alpha'} = \sum_i \alpha_i \cdot \frac{\partial S}{\partial \alpha_i}$$
Value-at-Risk

Value-at-Risk (VaR), a term and concept introduced by J. P. Morgan in the 1990’s and popularized by the Basle Committee as a certified measure of the risk in banks’ trading books.

\[ S(\alpha) = Q_{\psi_{\alpha}}(1 - c) \]

where the confidence level, \( c \), is usually 0.95 or 0.99.

Properties

Value-at-Risk has many properties that make it a candidate for an index of satisfaction, except

- not super-additive
- not concave
- not risk averse

Value-at-risk can be challenged by diversification. For example, a portfolio of defaultable bonds could have a lower satisfaction than a single defaultable bond.
Value-at-Risk

When it was first introduced by J. P. Morgan, it was argued that $M \sim \mathcal{N}(\mu, \Sigma)$ was an adequate description of risk in the market. We can recover a simple expression for the marginal decomposition in this setting.

$$Q_{\psi_\alpha}(1 - c) = \alpha' \cdot \left( \mu - \sqrt{2} \cdot \text{erf}^{-1}(2 \cdot c - 1) \cdot \frac{\Sigma \cdot \alpha}{\sqrt{\alpha' \cdot \Sigma \cdot \alpha}} \right)$$

A similar result holds generally.

$$Q_{\psi_\alpha}(1 - c) = \alpha' \cdot \mathbb{E} \left( M \mid \{ \alpha' \cdot M = Q_{\psi_\alpha}(1 - c) \} \right)$$

It is easy to interpret this in a simulation setting.

1. sample the market vector $N$ times and evaluate the objective in each
2. sort them in descending order and isolate a range of results around $c \cdot N$
3. the marginal value-at-risk for each position is the average in this range of the contribution
Beyond the normal approximation proposed by J. P. Morgan, the next most useful approximation to the quantile measure of satisfaction come from the Cornish-Fisher expansion.

**Cornish-Fisher Expansion**

In general,

\[
Q_{\psi_\alpha}(1 - c) = E(\psi_\alpha) \\
- Sd(\psi_\alpha) \cdot \left( z_1(c) - \frac{z_2(c) - 1}{6} \cdot Sk(\psi_\alpha) \right) + \cdots
\]

where \( z_1(c) = \sqrt{2} \cdot \text{erf}^{-1}(2 \cdot c - 1) \) and \( z_2(c) = z_1(c)^2 \).

In practice, the moments of \( \psi_\alpha \) can be evaluated from the Delta-Gamma approximation (\( * \)).
Expected Shortfall

The fact that quantile-based satisfaction can be challenged by diversification has led to a modified definition, expected shortfall or conditional value-at-risk.

\[
\text{ES}_c(\alpha) = \frac{1}{1 - c} \cdot \int_0^{1-c} Q_{\Psi_\alpha}(p) \, dp
\]

Spectral Index of Satisfaction

This is a special case of a more general definition,

\[
S(\alpha) = \int_0^1 \phi(p) \cdot Q_{\Psi_\alpha}(p) \, dp
\]

for the spectrum, \(\phi(\cdot)\), a decreasing function with

\[
\int_0^1 \phi(p) \, dp = 1 \text{ and } \phi(1) = 0.
\]

- Spectral satisfaction is super-additive, concave, and risk averse.
Expected Shortfall

The marginal decomposition is also similar to that of value-at-risk.

$$ES_c(\alpha) = \alpha' \cdot E \left( M \left| \{ \alpha' \cdot M \leq Q_{\Psi} \alpha (1 - c) \} \right. \right)$$

It is also subject to the same Cornish-Fisher expansion, with the replacement

$$\tilde{z}_1(c) = \frac{1}{1 - c} \cdot \int_c^1 z(p) \, dp$$

$$\tilde{z}_2(c) = \frac{1}{1 - c} \cdot \int_c^1 z(p)^2 \, dp$$

Contrasting

<table>
<thead>
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<th>$c$</th>
<th>$z_1(c)$</th>
<th>$z_2(c)$</th>
<th>$\tilde{z}_1(c)$</th>
<th>$\tilde{z}_2(c)$</th>
</tr>
</thead>
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<tr>
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<td>2.71</td>
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<td>2.33</td>
<td>5.41</td>
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<td>7.20</td>
</tr>
</tbody>
</table>

we see that expected shortfall is more sensitive to skewness in the objective then value-at-risk.
Expected Utility

A more traditional approach to defining investor satisfaction is to build an index of satisfaction based on utility, which is an economist’s approach to specifying preferences.

Certainty-Equivalent

We want our measure of investor satisfaction to be denominated in currency, while utility, $u(\Psi_\alpha)$, is denominated in “utils.” To resolve this, we can apply the inverse of utility to get back to units of the investor’s objective.

$$\text{CE}(\alpha) = u^{-1}(\mathbb{E}u(\Psi_\alpha))$$

Coherent Properties

Few general claims can be made. The power utility exhibits positive homogeneity and the exponential utility exhibits translation invariance. Generally,

- not super-additive, co-monotonic additive, or concave
Pearson Family

A useful three-parameter family of utility functions is given by the ordinary differential equation,

\[- \frac{u''(\psi)}{u'(\psi)} = \frac{\psi}{\gamma \cdot \psi^2 + \zeta \cdot \psi + \eta}\]

This includes the following classes

- **quadratic (CAPM)** $u(\psi) = \psi - \frac{1}{2\zeta} \cdot \psi^2$ for $\psi < \zeta$
- **log** $u(\psi) = \log \psi$ for $\psi > 0$
- **power** $u(\psi) = \psi^{1 - \frac{1}{\gamma}}$ for $\psi > 0$
- **exponential** $u(\psi) = -e^{-\frac{\psi}{\zeta}}$
- **prospect theory** $u(\psi) = \text{erf} \left( \frac{\psi}{\sqrt{2 \cdot \eta}} \right)$

Note that power and log utilities are only sensible if the support for the investor’s objective is bounded below, and the quadratic utility is only sensible if the support for the investor’s objective is *bounded above*. 
Arrow-Pratt Approximation

A power series expansion of the utility provides a useful approximation to the certainty-equivalent.

\[ CE(\alpha) \approx E\psi_\alpha + \frac{u''(E\psi_\alpha)}{2 \cdot u'(E\psi_\alpha)} \cdot \text{var } \psi_\alpha \]

This approximation is exact if the market vector is normal and the utility is exponential.

Risk Aversion

The certainty-equivalent satisfaction is invariant under changes of utility units. This motivates the definition of the Arrow-Pratt (absolute) risk aversion,

\[ A(\psi) = -\frac{u''(\psi)}{u'(\psi)} \]

To the extent that this is constant near the mean, the Arrow-Pratt approximation is quadratic in the allocation.
Prospect Theory

Prospect theory comes from psychological rather than traditional economic considerations. It is an example of behavioral economics.

- Properly, the investor’s objective is relative (either profit or profit relative to some benchmark)
- The investor becomes increasingly risk-averse with cumulative gains, but
- The investor becomes increasingly risk-seeking with cumulative losses

The Arrow-Pratt approximation for the certainty-equivalent satisfaction under the prospect theory utility is

\[ CE(\alpha) \approx E\psi_\alpha \cdot \left( 1 - \frac{1}{2 \cdot \eta} \cdot \text{var} \, \psi_\alpha \right) \]