Market Models
Practitioner Course: Interest Rate Models

John Dodson

March 29, 2009
Forward Measure

In order to value European-style options, we need to evaluate risk-neutral expectations of the form

\[ V(t, T) = E_t [D(t, T) \cdot H(T)] \]

where \( T \) is the exercise date, and the payoff is \( H(T) \), which is \( \mathcal{F}_T \)-measurable.

Terminal Correlation

When the payoff is interest-sensitive, evaluating this involves not only the expected values \( P(t, T) = E_t [D(t, T)] \) and \( E_t [H(T)] \) as in Black-Scholes analysis for stock options, but also the terminal correlation between the payoff and the stochastic discount factor.

\[
V(t, T) = P(t, T) \cdot E_t [H(T)]
+ \rho \cdot \sqrt{\text{var}_t [D(t, T)] \cdot \text{var}_t [H(T)]}
\]
Change of Measure

Alternatively, we can introduce a convenient change of measure, given by the Radon-Nikodym derivative

\[
\frac{dQ_t^T}{Q_t} = \frac{D(t, T)}{P(t, T)}
\]

whereby

\[
V(t, T) = P(t, T) \cdot E_t^T [H(T)]
\]

This is called the \( T \)-forward measure, which reflect the fact that \( E_t^T [r(T)] = f(t, T) \); i.e.

- the forward rate is the risk-neutral expectation of the spot rate under the \( T \)-forward measure.

In the Market Models, this change of measure is merely a formality, because we start off in the forward measure and never look back.
Simple Spot & Forward

For both practical and technical reasons, we need to start our treatment of the Market Models in a discrete setting. We have already defined the simple spot rate $L(t, T)$.

$$P(t, T) = \frac{1}{1 + \tau(t, T) \cdot L(t, T)}$$

Similarly, the simple forward rate between dates $T$ and $S > T$ is defined by

$$\frac{P(t, S)}{P(t, T)} = \frac{1}{1 + \tau(T, S) \cdot F(t; T, S)}$$

and $F(T; T, S) = L(T, S)$
Payment in Arrears

Consider a European-style derivative with exercise date $T$ and an $\mathcal{F}_T$-measurable payoff $H(T) = h(L(T, S))$ which pays in arrears on date $S$. The value of this derivative today is

$$V(t, T, S) = E_t [D(t, S) \cdot h(L(T, S))]$$

$$= P(t, S) \cdot E_t^S [h(F(T; T, S))]$$  \hspace{1cm} (1)$$

In order to evaluate (1), we need to first specify the process for $F(t; T, S)$ under the $S$-forward measure.
Discrete Forward Rates

Lognormal Forward Rates

Since we can write $F(t; T, S)$ as the value of an asset in terms of the numéraire associated with the $S$-forward measure $P(t, S)$,

$$F(t; T, S) = \frac{\frac{1}{\tau(T,S)} \cdot P(t, T) - \frac{1}{\tau(T,S)} \cdot P(t, S)}{P(t, S)}$$

we know that $F(t, T, S)$ must be a martingale under the $S$-forward measure. For example, we could impose

$$dF(t; T, S) = \sigma_S(t) \cdot F(t; T, S) \ dW^S \quad \forall \quad t \leq T \quad (2)$$

ensuring that the forward rate is non-negative and that the evolution is Markovian.
Lognormal Forward Rates

The model in (2) can be integrated to yield

$$\log L(T, S)|\mathcal{F}_t$$

$$\sim N \left( \log F(t, T, S) - \frac{1}{2} \int_t^T \sigma_S^2(t') \, dt', \int_t^T \sigma_S^2(t') \, dt' \right)$$

under the $S$-forward measure.

Black Formulæ

And since the payoffs for caplets and floorlets are of the form $h(L(T, S)) = \tau(T, S) \cdot (L(T, S) - K)^+$ reminiscent of calls and puts, the lognormal process leads directly to the Black formulæ.
Caps & Floors

A cap (floor) is just a portfolio of caplets (floorlets). Each can be valued separately under the corresponding forward measure. Consider a sequence of fixing and payment dates, \( \mathcal{T} = \{ T_0 = t, T_1, \ldots, T_n = T \} \). The value of a cap is

\[
V_{\text{cap}}(t, \mathcal{T}, K) = \sum_{i=1}^{n} P(t, T_i) \cdot \tau(T_i, T_{i-1}) \cdot E_{t}^{T_i} (L(T_{i-1}, T_i) - K)^+
\]

or, substituting in the Black formula in the obvious fashion,

\[
V_{\text{cap}}(t, \mathcal{T}, K) = \sum_{i=1}^{n} P(t, T_i) \cdot \tau(T_i, T_{i-1}) \cdot \text{BI}_{\text{call}} \left( K, F(t, T_{i-1}, T_i), \sqrt{\int_t^{T_{i-1}} \sigma_{T_i}^2(t') \, dt'} \right)
\]
Caps & Floors

Cap Volatility

Typically the market will quote a sort-of average implied Black volatility $\sigma_T$ solving

$$V_{\text{cap}}(t, T, K) = \sum_{i=1}^{n} P(t, T_i) \cdot \tau(T_i, T_{i-1}) \cdot \text{BI}_{\text{call}} \left( K, F(t, T_{i-1}, T_i), \sigma_T \cdot \sqrt{T_{i-1} - t} \right)$$

and it will be left to the modeler to determine how to strip these into the individual volatilities of the forward rates.

- Here, the different measures and potential correlations do not present a problem;
- but parsing out the time-dependence between the different forward rates require will require modeling choices as in §6.3.1.
Instantaneous Forward Rate Volatility

Let us briefly review the calibration problem for caps & floors. Let us start by assuming that each forward rate is piece-wise constant. That is,

$$\int_{t}^{T_{i-1}} \sigma_{T_{i}}^{2}(t') \, dt' = \sum_{j=1}^{i-1} \sigma_{i,j}^{2} \cdot (T_{j} - T_{j-1})$$

If we have data on $N$ different caps, there are $N$ different forward rate and $\frac{1}{2}N \cdot (N + 1)$ possible instantaneous volatilities.

<table>
<thead>
<tr>
<th></th>
<th>$(t, T_{1}]$</th>
<th>$(T_{1}, T_{2}]$</th>
<th>...</th>
<th>$(T_{N-1}, T_{N}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(T_{1}, T_{2})$</td>
<td>$\sigma_{1,1}$</td>
<td>n/a</td>
<td>...</td>
<td>n/a</td>
</tr>
<tr>
<td>$F(T_{2}, T_{3})$</td>
<td>$\sigma_{2,1}$</td>
<td>$\sigma_{2,2}$</td>
<td></td>
<td>n/a</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>n/a</td>
</tr>
<tr>
<td>$F(T_{N}, T_{N+1})$</td>
<td>$\sigma_{N,1}$</td>
<td>$\sigma_{N,2}$</td>
<td>...</td>
<td>$\sigma_{N-1,N}$</td>
</tr>
</tbody>
</table>

In practice, we will need to settle on a scheme to reduce this count to $O(N)$ or ideally $O(1)$.
Since

1. caps tell us nothing about the (instantaneous) correlations amongst rates, and since

2. we have so far only been able to value swaptions using the Jamshidian decompositions with its assumption of perfect correlations

The LFM would seem to offer little help in valuing swaptions. But recalling an important result about the value of a swap

\[ V_{\text{swap}}(t, T, K) = (K - s(t, T)) \cdot \int_t^T P(t, T') d\tau(T') \]

gives us a way to proceed in the spirit of the LFM, which we will call the lognormal swap model (LSM).
Consider a European-style derivative with exercise date $T$ and $\mathcal{F}_t$-measurable payoff $H(T)$ which depends on the swap rate $s(T, S)$ for some term $S > T$ according to $H(T) = h(s(T, S))$. Under the change of measure

$$\frac{d\mathbb{Q}^T_{t,S}}{d\mathbb{Q}_t} = \frac{B(t)}{B(T)} \cdot \frac{\int_T^S P(T, T') \, d\tau(T')}{\int_T^S P(t, T') \, d\tau(T')},$$

we have that the value of the derivative is

$$V(t, T) = \mathbb{E}_t \left[ D(t, T) \cdot H(T) \cdot \int_T^S P(T, T') \, d\tau(T') \right] = \int_T^S P(t, T') \, d\tau(T') \cdot \mathbb{E}_{t}^{T,S} \left[ h(s(T, S)) \right]$$
Lognormal Forward Swap Rates

Furthermore, the forward swap rate $s(t; T, S)$ is a martingale under $\mathbb{Q}^{T,S}_t$. We can see this because the forward swap rate is the value of an asset in terms of the new forward annuity factor numéraire.

$$s(t; T, S) = \frac{P(t, T) - P(t, S)}{\int_T^S P(t, T') \, d\tau(T')}$$

Lognormal Forward Swap Rates

Therefore, in continuing analogy to the LFM we can impose a process such as

$$ds(t; T, S) = \sigma_{T,S}(t) \cdot s(t; T, S) \, dW^{S,T} \quad \forall \quad t \leq T$$

with $s(T; T, S) = s(T, S)$. 
Lognormal Forward Swap Rates

Again this can be integrated to yield

\[ \log s(T, S) | \mathcal{F}_t \]
\[ \sim N \left( \log s(t, T, S) - \frac{1}{2} \int_t^T \sigma_{T,S}(t') dt', \int_t^T \sigma_{T,S}^2(t') dt' \right) \]

under the \( T, S \)-forward swap measure.

Swaptions

For example, we can immediately get results for swaptions values.

\[ V_{RTR}(t, T, S, K) = \int_T^S P(t, T') d\tau(T') \cdot \text{BL}_{\text{put}} \left( K, s(t; T, S), \sqrt{\int_t^T \sigma_{T,S}^2(t') dt'} \right) \]
Swaption Volatility

Translations
The LFM and LSM are fundamentally incompatible, and neither are compatible with tractable short-rate models.

- There are approximate schemes for translating volatilities, such as Rebonato’s formula.

But forward rate volatilities alone are not sufficient to value swaptions, because caps tell us nothing about correlations. Additional complexity is also apparent when we consider that if there are $N$ reset dates, there are potentially $\frac{1}{2} N \cdot (N - 1)$ swaptions.

Correlations
So even if we are successful in fitting instantaneous forward volatilities with $O(N)$ or $O(1)$ parameters,

- we still need to deal with potentially $\frac{1}{2} N \cdot (N - 1)$ instantaneous correlations

§6.9 discusses various schemes for this.