Market Models
MFM Practitioner Module:
Risk & Asset Allocation

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Outline

Modeling the Market
  Securities Markets
  Market Conventions
  Investment Horizon

Quest for Invariance
  Identifying Invariants
  Projecting Invariants
  Mapping Invariants
Securities

Securities are claims on future cashflows from their issuer.

- **U. S. Treasury**
  - (nominal, indexed) bond
- **bank**
  - interbank loan/deposit, commercial paper, repo
  - swap, over-the-counter derivative, currency contract
- **corporation**
  - (common, preferred) equity
  - (secured, senior, subordinated, convertible) bond
  - commercial paper
- **municipality**
  - (revenue, general obligation) bond
- **derivatives clearinghouse**
  - futures, option
- **collective investments**
  - (open-ended, closed-ended, exchange-traded) funds and unit trusts

**Also:** real estate, private equity, bank loan, etc.
Securities Conventions

- equity trades shares and **lots** of 100 shares, and pays **dividends** to the registered holders as of the **ex date**
  - price quotation is per share, and trades settle in about three business days
  - the **broker** may be able to provide financing or locate shares to **short**
- bonds trade in increments of $1,000 **par amount** and pay periodic (annual or semi-annual) **coupons**
  - price quotation is per $100 notional and excludes **accrued interest** for the current coupon
  - settlement is typically two business days
- futures settle daily through a **margin account** according to the **tick size** and the **settlement price**
- the **underlying** for equity options is typically 100 shares, and they settle upon **exercise**
- public open-ended funds trade at the end-of-day **net asset value** per share; ETF’s trade like equities
**Investment Horizon**

**Single Horizon**

Meucci represents the market by

\[ P_{T+\tau} \in \left( \mathbb{R}^+ \right)^N \]

which is a random variable representing the unknown values of \( N \) securities \( \tau \) years into the future from the decision date \( T \).

**Inter-period Outcomes**

Security prices are modeled elsewhere by stochastic processes, recognizing that prices evolve through time with the flow of information. The value of a portfolio also evolves through time.

- But here we only consider discrete portfolio allocation decisions, which have been scheduled in advance.
Choosing an Investment Horizon

Choosing an investment horizon for optimization involves weighing trade-offs

- How much effort does it take to re-calibrate the models and re-run the optimization?
- How much will it cost over time to change the allocations, in terms of transactions costs?
- How much is the outcome from the current portfolio likely to differ from the outcome from the optimal portfolio?
- Can I or should I re-balance the portfolio in the mean-time?
- Under what circumstances would I feel obliged to re-allocate the portfolio early?
“Quest for Invariance”

1. detecting invariants
2. determining the distribution of invariants
3. projecting the distribution into the future
4. mapping the projections back into simulated market prices

Some Challenges

- serial autocorrelation
  - stale prices
  - non-simultaneous quotations
- time inhomogeneity
  - conditional heteroskedasticity
  - seasonality
Building the Dataset

We will want to identify a function $g$ and a random variable $X$ such that:

1. The projection $P_{T+\tau} = g(X; p_T)$
2. Observations of $X$ can be derived from historical observations of the market, $\{p_T, p_{T-\tau}, p_{T-2\tau}, \ldots\}$, which we can believe are independent and identically distributed (i.i.d.)
3. We know the characterization of $X$ up to a finite number of parameters, which we can estimate from the dataset above.

N.B.: Meucci leaves it unmentioned, but the market history needs to properly account for corporate actions such as:

- coupon, principal, or dividend payments; and
- share splits, spin-offs, and mergers.
Identifying Invariants

Meucci gives examples of plausible invariants for various markets:

- equities, commodities, currencies: total return
- bonds: changes in discount rates and spreads
- options: changes in at-the-money implied volatilities

Total Return

Total return, which we will talk about in the next slides, is appropriate for equity shares under the assumption that the issuer is an on-going concern, but it is not an appropriate candidate for bonds or derivatives which eventually mature or expire.
The efficient markets hypothesis suggests that

\[ H_{t,\tilde{\tau}} \overset{\text{def.}}{=} \frac{P_t + \int_{t-\tilde{\tau}}^{t} e^{r(t'-t)} dK_{t'}}{P_{t-\tilde{\tau}}} \]

where \( P \) denotes the price per share and \( K \) represents the cumulative cashflow per share, should be independent over time for non-overlapping periods for equity shares.

- Otherwise, there would be arbitrage, according to the Efficient Market Hypothesis.
- The stronger assumption that Meucci makes here is that these are \textit{identically distributed}.
Total Return

There are two common representations of this quantity

- simple return \( L = H - 1 \)
- compound return \( C = \log H \approx L - \frac{1}{2} \cdot L^2 \)

both of which are expected to be approximately symmetric about zero. In these terms, the invariant is a vector random variable \( X \) whose characterization has been calibrated by the datasets \( \{H_t\}, \{L_t\}, \text{or } \{C_t\}; \) and the mapping is

\[
P_{T+\tau} = g(X; p_T) = p_T \cdot X
= p_T \cdot (1 + X)
= p_T \cdot e^X
\]

component-wise respectively.
Implied Volatility

Options as investments carry exposure to changes a number of factors, including

- the spot price of the underlying
- dividends between now and expiration
- financing rates
- implied volatility

Since options values are co-monotonic with the value of the underlying and a risk-free loan, changes in the implied volatility is a candidate for an invariant.

**Implied Volatility**

For an at-the-money Euro. call, we have the relationship

\[ \sigma_t = \sqrt{\frac{8}{T - t}} \operatorname{erf}^{-1} \left( \frac{c_t}{F_t e^{-r(T-t)}} \right) \approx \sqrt{\frac{2\pi}{T - t}} \frac{c_t}{P_t} \]
Projecting the Invariants

Time Scale
We need to consider the case that the historical sampling period, \( \tilde{\tau} \), is not equal to the investment horizon, \( \tau \). For example, say we have a characterization of daily data, \( Y \), but we plan to re-allocate only monthly, based on \( X \).

Exercise
If the investment horizon, \( \tau \), is a multiple of the sampling period \( \tilde{\tau} \), it is natural to define \( X \) as the sum of \( \frac{\tau}{\tilde{\tau}} \) independent draws of \( Y \).

1. If the observations \( Y_i \) are i.i.d., what is the characteristic function for \( X \) in terms of \( \phi_Y(t) \)?
2. In terms of this result, what is \( \mathbb{E} \left[ X \right] = -i \cdot \frac{d\phi_X}{dt'} \bigg|_0 \)?
3. What is \( \mathbb{E} \left[ X \cdot X' \right] \)?
4. What is the relationship between \( \text{cov} X \) and \( \text{cov} Y \)?
Mapping Back to the Market

The fact that $P_{T+\tau} > 0$ suggests a general functional form for the mapping $g$ between the market invariants and security prices.

$$P_{T+\tau} = g(X; p_T) = e^{\gamma + \text{diag} \epsilon \cdot X}$$

where $\gamma$ and $\delta$ are constant vectors and the exponential is taken component-wise.

Example: Total Return

The total return case has a particularly simple representation in this form.

$$\gamma = \log p_T$$

$$\epsilon = 1$$

Other common market invariants, such as discounting rates, can also be expressed.
Mapping Invariants

This is a simple enough transformation to evaluate the complete density of the result,

\[ f_P(P) = \frac{f_X \left( (\text{diag } \epsilon)^{-1} \cdot (\log P - \gamma) \right)}{\prod_n \epsilon_n \cdot P_n} \]

but if we are interested in just location and dispersion, the characteristic function gives us a short-cut.

\[ E(P_m \cdot P_n) = e^{\gamma_m + \gamma_n} \cdot \phi_X \left( -i \cdot \epsilon_m \cdot \delta^{(m)} - i \cdot \epsilon_n \cdot \delta^{(n)} \right) \]

where \( \delta^{(n)} \) is the \( n \)-th basis vector, (\( \ldots, 0, 1, 0, \ldots \)\)'