Mean-Variance Optimization

MFM Practitioner Module: Risk & Asset Allocation

John Dodson

January 27, 2010
Constraints

Optimization
  Dimension Reduction
  Analytical Solution
  Log-Normal Model
  Time Scaling
  Benchmarks
Optimizing Allocations

Once we have
1. chosen the markets and an investment horizon
2. modeled the markets
3. agreed on an objective with the client
4. designed an index of satisfaction

We are almost ready to determine the optimal portfolio

Constraints
The last item we need to consider before proceeding with the mechanical exercise of optimization is the

- domain of possible allocations, $C$

Wealth Constraint
One constraint that almost always applies is the wealth constraint

$$C = \{ \alpha \ni p_T' \cdot \alpha = w_T \}$$

for given initial wealth $w_T$. 
Secondary Constraints

In addition to or instead of the wealth constraint, other constraints may be relevant

- transactions costs
- exogenous risk measures and limits
- prohibition against short positions

Each binding constraint has a corresponding shadow cost reflecting the fact that the otherwise unconstrained optimal portfolio is not feasible.

- Professional investment management is largely about managing constraints.

N.B.: The form and complexity of the constraints determines the form and complexity of the optimization techniques required.
Optimization Problem

The optimal portfolio, if it exists, is defined by

\[ \alpha^* = \arg \max_{\alpha \in C} S(\alpha) \]

Constrained optimization in general is a hard problem. There are several classes that are more tractable.

**Convex Programming**

If the objective \( Q \) is convex, and the constraints can be represented by the intersection of sets \( L = \{ z \in A \cdot Z = a \} \) and \( V = \{ z \in F(z) \leq 0 \} \) for convex \( F(\cdot) \), the problem

\[ \arg \min_{z \in L \cap V} Q(z) \]

has a solution, although it may be difficult or expensive to determine.

- This is why we should choose an index of satisfaction that is concave.
If the index of satisfaction satisfies stochastic dominance, we may be able to replace the original hard problem with two simpler problems.

**Stochastic Dominance**

An allocation $\alpha$ is said to weakly dominate an allocation $\beta$ if

$$F_{\psi \alpha}(\psi) \leq F_{\psi \beta}(\psi) \quad \forall \psi \text{ in support}$$

and an index of satisfaction is consistent with weak stochastic dominance if $S(\alpha) \geq S(\beta)$.

- value-at-risk and expected shortfall are both consistent with weak stochastic dominance

In particular, if $\text{Sd } \Psi_{\alpha} = \text{Sd } \Psi_{\beta}$ but $\mathbb{E}\Psi_{\alpha} > \mathbb{E}\Psi_{\beta}$ then $S(\alpha) > S(\beta)$
Dimension Reduction

Two-Step Procedure

As long as the index of satisfaction is consistent with weak stochastic dominance, we can exclude from consideration in the search for the optimal portfolio any portfolio that is not on the efficient frontier, defined by

1. \( \alpha^*(v) = \arg \max_{\alpha \in C} \frac{\mathbb{E} \Psi_{\alpha}}{\var \Psi_{\alpha} \leq v} \)

The optimal portfolio is therefore the solution to a one-dimensional unconstrained problem,

2. \( v^* = \arg \max_v S(\alpha^*(v)) \)

which can be found numerically, viz.

\[ \alpha = \alpha^*(v^*) \]

This framework presents an enormous computational improvement, and justifies choosing an index of satisfaction that is consistent with weak stochastic dominance.
Analytical Solution

For a single, affine constraint, such as the wealth constraint, the efficient frontier can be derived analytically. First we introduce the dual formulation

$$\alpha^*(v) = \arg \min_{d' \cdot \alpha = c} \alpha' \cdot \text{cov} \ M \cdot \alpha$$

subject to

$$\alpha' \cdot \text{E} M \geq e$$

The constraints can be handled by introducing two (scalar) Lagrange multipliers, $\lambda$ and $\mu$.

$$\min_{\alpha, \lambda, \mu} \alpha' \cdot \text{cov} \ M \cdot \alpha - \lambda \cdot (\alpha' \cdot d - c) - \mu \cdot (\alpha' \cdot \text{E} M - e)$$

Any solution to this (see Appendix 6.3) can be expressed as a linear combination of two portfolios,

$$\alpha_{SR} = c \frac{(\text{cov} \ M)^{-1} \cdot \text{E} M}{d' \cdot (\text{cov} \ M)^{-1} \cdot \text{E} M}$$

$$\alpha_{MV} = c \frac{(\text{cov} \ M)^{-1} \cdot d}{d' \cdot (\text{cov} \ M)^{-1} \cdot d}$$
Analytical Solution

The efficient frontier consists of all portfolios

\[(1 - \beta)\alpha_{MV} + \beta\alpha_{SR} \quad \forall \beta \geq 0\]

N.B.: \(\alpha_{MV}\) and \(\alpha_{SR}\) are orthogonal, in the sense that
\[\text{cov} (\Psi_{\alpha_{SR}}, \Psi_{\alpha_{MV}}) = 0\]

Sharpe Ratio Portfolio

The portfolio \(\alpha_{SR}\) is the unique portfolio that maximizes the Sharpe Ratio.

\[\alpha_{SR} = \arg \max d' \cdot \alpha = c \frac{\mathbb{E} \Psi_{\alpha}}{\text{std} \Psi_{\alpha}}\]

Minimum Variance Portfolio

The portfolio \(\alpha_{MV}\) is the globally minimum variance portfolio within the feasible set. If there is an investment that has a risk-free objective, e.g. cash or a benchmark-neutral allocation, then it will consist of this.
Log-Normal Model

The first step of the two-step procedure requires mean and covariance of the market vector, regardless of the form of the index of satisfaction. Therefore, let us review these quantities for the standard log-normal model

\[ P_{T+\tau} = p_T \times e^X \quad \text{with} \quad X \sim \mathcal{N}(\mu, \Sigma) \]

where the exponentiation and \( \times \) is component-wise.

Recalling that \( \phi_X(t) = e^{i \cdot \mu' \cdot t - \frac{1}{2} \cdot t' \cdot \Sigma \cdot t} \), we can get

**Mean**

\[ \mathbb{E} P_{T+\tau} = p_T \times e^{\mu + \frac{1}{2} \cdot \text{diag} \Sigma} \]

**Covariance**

\[ \text{cov} \ P_{T+\tau} = \mathbb{E} P_{T+\tau} \cdot \mathbb{E} P'_{T+\tau} \times \left( e^{\Sigma} - 1 \cdot 1' \right) \]
Log-Normal Model

The author points out that the efficient frontier depends in a non-trivial way upon the investment horizon, \( \tau \).

Time Scaling

Say you had a model for the invariants \( X \sim \mathcal{N}(\mu, \Sigma) \) based on a sampling period of \( \tau \), but you were interested in a different investment horizon \( \tilde{\tau} \). We know that \( X \) is additive, while \( e^X \) is not. In particular,

\[
E P_{T+\tilde{\tau}} = p_T \times e^{\frac{\tilde{\tau}}{\tau}(\mu + \frac{1}{2} \text{diag} \Sigma)}
\]

\[
\text{cov} \ P_{T+\tilde{\tau}} = E P_{T+\tilde{\tau}} E P'_{T+\tilde{\tau}} \times \left( e^{\frac{\tilde{\tau}}{\tau} \Sigma} - 1 \cdot 1' \right)
\]

which leads to a different solutions for \( \alpha_{SR} \) and maybe \( \alpha_{MV} \).

This is not apparent in the traditional analysis in terms of continuous returns and underscores how important it is to choose an appropriate horizon.
Benchmarks

When the investor’s objective is relative, the wealth constraint is of the form $d' \cdot \alpha = 0$.

- In this case, the minimum variance portfolio is trivial, $\alpha_{MV} = 0$; but
- the solution for the maximum Sharpe Ratio portfolio fails (since $c = 0$).

In place, we can use

$$\alpha_{SR} = (\text{cov } M)^{-1} \cdot \left( d' \cdot (\text{cov } M)^{-1} \cdot d \cdot E M - d' \cdot (\text{cov } M)^{-1} \cdot E M \cdot d \right)$$

and all efficient portfolios have the same Sharpe Ratio (termed the Information Ratio in the relative setting).

Implied Benchmark

The efficient frontier for the absolute objective is the same as the efficient frontier for the objective relative to the global minimum-variance portfolio $\alpha_{MV}$. 