1 Problem

In this week’s case discussion, we saw that the estimator and standard error from ordinary least squares for linear regression is equivalent to the conditional expectation and variance of the dependent variable conditional on an observation of the independent variable(s) under the assumption that unconditionally they are jointly normal.

You are asked to use mode and modal dispersion in place of expectation and variance to re-formulate this (for a single dependent and a single independent variable) for random variables that are instead jointly Cauchy.

2 Solution

From Meucci (2.209) with $N = 2$ and a bit of algebra, we can write the density of a bivariate Cauchy r.v.

$$f_{X,Y}(x, y) = \frac{1 - \rho^2}{2\pi\sigma_X\sigma_Y} \left( 1 - \rho^2 + \left( \frac{x - \mu_X}{\sigma_X} \right)^2 - 2\rho \cdot \left( \frac{x - \mu_X}{\sigma_X} \right) \left( \frac{y - \mu_Y}{\sigma_Y} \right) + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 \right)^{-3/2}$$

where we have written the dispersion matrix (without loss of generality) as

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{pmatrix} \quad \text{hence} \quad \Sigma^{-1} = \frac{1}{\sigma_X^2\sigma_Y^2 (1 - \rho^2)} \begin{pmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_Y\sigma_X & \sigma_X^2 \end{pmatrix}$$

The question asks us to compare the marginal distribution of $Y$ to the conditional distribution of $Y|\{X = x\}$.

The marginal distribution of $Y$ is just a univariate Cauchy.

$$f_Y(y) = \frac{1}{\pi\sigma_Y} \frac{1}{1 + \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2}$$

which has mode $\mu_Y$ and modal dispersion $\frac{1}{2}\sigma_Y^2$. 

The conditional distribution works out to be

\[
f_{Y \mid X = x}(y) = \frac{1}{2\sigma_Y \sqrt{1 - \rho^2}} \left( 1 + \frac{y - \left( \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X) \right)}{\sigma_Y \sqrt{1 - \rho^2} \sqrt{1 + \left( \frac{x - \mu_X}{\sigma_X} \right)^2}} \right)^{-3/2}
\]

which has a mode \( \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X) \) and modal dispersion \( \frac{1}{3} \sigma_Y^2 \left( 1 - \rho^2 \right) \left( 1 + \left( \frac{x - \mu_X}{\sigma_X} \right)^2 \right) \).

Notice the conditional distribution is not in the Cauchy family.

Observing \( \{X = x\} \) should change our opinion about \( \{Y = y\} \) according to

\[
\text{Mode } [Y \mid X = x] = \text{Mode } [Y] + \rho \sqrt{\frac{\text{MDis } [Y]}{\text{MDis } [X]}} (x - \text{Mode } [X])
\]

\[
\text{MDis } [Y \mid X = x] = \text{MDis } [Y] \left( 1 - \rho^2 \right) \left( \frac{2}{3} + \frac{1}{3} \left( x - \text{Mode } [X] \right)^2 \right)
\]

The conditional location result is analogous to that from the bivariate normal case. But the conditional dispersion result is notably different. Not only does it depend on the level of the independent variable, but there are situations in which the observation can actually lead to an increase in the uncertainty.