Say \( X \) is the random variable that will drive market prices between today, time \( T \), and the next decision date, time \( T + \tau \) with an investment horizon of \( \tau \) (all time measured in years).

\[
P_{T+\tau} = g(X; P_T)
\]

and say that we have used historical data to estimate the parameters of the characterization of the market invariants \( Y \) under a different horizon, \( \tilde{\tau} \); e.g., from a timeseries sample \( \{P_T, P_{T-\tilde{\tau}}, P_{T-2\tilde{\tau}} \ldots \} \).

We know from the properties of characteristic functions that as long as increments are independent and identically distributed,

\[
\phi_X(t) = \phi_Y(t)^{\tilde{\tau}}
\]

If the first two moments exist, we also know that

\[
\begin{align*}
E(X) &= -i \left. \frac{d\phi_X}{dt'} \right|_0 \\
&= -i \left. \frac{\tau}{\tilde{\tau}} \phi_Y^{\tilde{\tau}-1} \frac{d\phi_Y}{dt'} \right|_0 \\
&= \frac{\tau}{\tilde{\tau}} E(Y)
\end{align*}
\]

and

\[
\begin{align*}
E\left(XX'\right) &= - \left. \frac{d^2\phi_X}{dt'dt} \right|_0 \\
&= -\left(\frac{\tau}{\tilde{\tau}} - 1\right) \left(\phi_Y\right)^{\tilde{\tau}-2} \left. \frac{d\phi_Y}{dt'} \frac{d\phi_Y}{dt} \right|_0 - \left(\frac{\tau}{\tilde{\tau}}\right)^{\tilde{\tau}-1} \left. \frac{d^2\phi_Y}{dt'dt} \right|_0 \\
&= \frac{\tau}{\tilde{\tau}} \left(\frac{\tau}{\tilde{\tau}} - 1\right) EYY' + \frac{\tau}{\tilde{\tau}} E(YY') \\
&= E\left(X\right)E\left(X'\right) + \frac{\tau}{\tilde{\tau}} \left(E(YY') - EYEY'\right)
\end{align*}
\]
Since the covariance is defined as
\[ \text{cov} Y = \text{E} \left( XX' \right) - \text{E} X \text{E} X' \]
we have that
\[ \text{cov} X = \frac{\tilde{\tau}}{\tilde{\tau}} \text{cov} Y \]
Furthermore, since
\[ \text{std} Y = \text{diag} \sqrt{\text{diag} \text{diag} \text{cov} Y} \]
we have the “square-root rule” for time-scaling market invariants.
\[ \text{std} X = \sqrt{\frac{\tilde{\tau}}{\tilde{\tau}}} \text{std} Y \]
This is valid regardless of the distribution of \( Y \) (as long as it has two moments).
Note that in general \( X \) will not belong to the same family of random variables as \( Y \).