Let us consider the expected shortfall index of satisfaction for a very simple portfolio: $\alpha$ shares in an asset whose value today is $p > 0$ and whose horizon value $P$ is lognormal.

Let us assume that the objective measure is profit; therefore in Meucci’s notation, we have

$$
\Psi_\alpha = \alpha M \\
= \alpha (P - p) \\
= \alpha (g(X) - p) \\
= \alpha p (e^X - 1)
$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean $\mu$ and variance $\Sigma > 0$. The index of satisfaction is

$$
S(\alpha) = \frac{1}{1-c} \int_0^{1-c} Q_{\Psi_\alpha}(q) \, dq
$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

## 1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.
We proceed to evaluate the exact version by considering the CDF of the objective.

\[ F_{\Psi_\alpha}(z) = P \{ \Psi_\alpha < z \} \]

\[ = P \left\{ \alpha p (e^X - 1) < z \right\} \]

\[ = P \left\{ X \, \text{sgn } \alpha < \log \left( 1 + \frac{z}{\alpha p} \right) \, \text{sgn } \alpha \right\} \]

\[ = P \left\{ \frac{X - \mu}{\sqrt{\Sigma}} \, \text{sgn } \alpha < \log \left( 1 + \frac{z}{\alpha p} \right) - \frac{\mu}{\sqrt{\Sigma}} \, \text{sgn } \alpha \right\} \]

\[ = \Phi \left( \log \left( 1 + \frac{z}{\alpha p} \right) - \frac{\mu}{\sqrt{\Sigma}} \, \text{sgn } \alpha \right) \]

where \( \Phi(\cdot) \) is the CDF of a standard normal.

The quantile, which is the inverse of the CDF, is therefore

\[ Q_{\Psi_\alpha}(q) = \alpha p \left( e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) \]

So can proceed to evaluate the index of satisfaction.

\[ S(\alpha) = \frac{1}{1 - c} \int_0^{1-c} \alpha p \left( e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) \, dq \]

\[ = \alpha p \left( \frac{1}{1 - c} \int_0^{1-c} e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} \, dq - 1 \right) \]

\[ = \alpha p \left( \frac{1}{1 - c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} z} \phi(z) \, dz - 1 \right) \]

where the last line is achieved by the change of variable \( z = \Phi^{-1}(q) \) and \( \phi(z) = \Phi'(z) \)

is the density of a standard normal.

Since

\[ e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} z} \phi(z) = e^{\mu + \frac{1}{2} \Sigma} \phi \left( z - \text{sgn } \alpha \sqrt{\Sigma} \right) \]

we have the final result,

\[ S(\alpha) = \alpha p \left( e^{\mu + \frac{1}{2} \Sigma} \frac{\Phi^{-1}(1-c) - \text{sgn } \alpha \sqrt{\Sigma}}{1 - c} - 1 \right) \quad (1) \]

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

\[ S(\alpha) \approx \alpha p \left( \mu - \text{sgn } \alpha \frac{\phi \left( \Phi^{-1}(1-c) \right) }{1 - c} \sqrt{\Sigma} \right) \quad (2) \]
Let us spend a moment interpreting this. An investor will be more satisfied to be long \( (\alpha > 0) \) if the asset has a positive expected return \( (\mu > 0) \), and short \( (\alpha < 0) \) if the asset has a negative expected return \( (\mu < 0) \). In contrast, positive variance diminishes satisfaction for any non-zero position.

This all seems quite reasonable for a rational index of satisfaction.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the objective \( \Psi_\alpha \). In a Delta-Gamma setting, we can replace the objective by the quadratic

\[
\Psi_\alpha = \alpha p \left( e^X - 1 \right) \approx \alpha p \left( X + \frac{1}{2} X^2 \right)
\]

hence \( \Theta_\alpha = 0 \), \( \Delta_\alpha = \alpha p \), and \( \Gamma_\alpha = \alpha p \). Let us define a new objective to represent this approximation.

\[
\Xi_\alpha = \alpha p \left( X + \frac{1}{2} X^2 \right)
\]

Is is straight-forward to work out that the first several central moments of this are

\[
\begin{align*}
E(\Xi_\alpha) &= \alpha p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) \\
Sd(\Xi_\alpha) &= |\alpha| p \sqrt{\Sigma} \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} \\
Sk(\Xi_\alpha) &= 3 \text{sgn} \alpha \sqrt{\Sigma} \left( \frac{(1 + \mu)^2 + \frac{1}{2} \Sigma}{(1 + \mu)^2 + \frac{1}{2} \Sigma} \right)^{3/2}
\end{align*}
\]

The third-order Cornish-Fisher expansion for expected shortfall in general is

\[
S(\alpha) \approx E(\Xi_\alpha) + Sd(\Xi_\alpha) \left( z_1 + \frac{z_2 - 1}{6} \ Sk(\Xi_\alpha) \right)
\]
with coefficients

\[
\begin{align*}
    z_1 &= \frac{1}{1-c} \int_{0}^{1-c} \Phi^{-1}(q) \, dq = -\frac{\phi \left( \Phi^{-1}(1-c) \right)}{1-c} \\
    z_2 &= \frac{1}{1-c} \int_{0}^{1-c} \Phi^{-1}(q)^2 \, dq = 1 - \frac{\phi \left( \Phi^{-1}(1-c) \right)}{1-c} \Phi^{-1}(1-c)
\end{align*}
\]

depending on the confidence level \( c < 1 \).

Putting this together, we get a third expression for the index of satisfaction.

\[
S(\alpha) \approx \alpha p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) - |\alpha| p \frac{\phi \left( \Phi^{-1}(1-c) \right)}{1-c} \sqrt{\Sigma} \cdot \left( \sqrt{(1+\mu)^2 + \frac{1}{2} \Sigma} + \frac{1}{2} \text{sgn} \alpha \left( \frac{1+\mu)^2 + \frac{1}{2} \Sigma \Phi^{-1}(1-c) \sqrt{\Sigma} \right) \right)
\]

This result agrees with (2) to lowest order in \( \mu \) and \( \Sigma \).

4 Exercise

Our horizon asset value \( P \) is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

\[
\Psi_\alpha = \alpha p \left( He^X - 1 \right)
\]

where \( X \sim N(\mu, \Sigma) \) as before\(^2\), but now we add an independent default indicator \( H \sim \text{Bern}(1-q) \) for default probability \( q \).

It is a worthwhile exercise to work out the three versions of the satisfaction under this new market model and compare them with their non-defaultable analogs.

\(^1\)The trick to these integrals is to realize that \( \phi'(z) = -z\phi(z) \).

\(^2\)Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.