Practical Topics in Optimization

MFM Practitioner Module: Risk & Asset Allocation

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Outline

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Cash

On the one hand, “cash” is just another asset class in your investment universe

▶ Consider, for example, a portfolio that can invest in assets traded in several currencies. What do we even mean by cash in this case?

But on the other, the investor’s objective is measured in one particular currency, and the risk-free investment in that currency has some unique properties making it worthy of exceptional treatment.

Analytical Solution

If you are using the analytical solution to the efficient frontier, the MV portfolio will have only cash, and the SR portfolio will have no cash. So you probably do not need to include cash explicitly in your market vector.
If you must include cash in your model for the market vector $M$, be careful that it does not make $\text{cov } M$ singular.

**Implicit Cash**

If you are able to avoid including it in the model for $M$, make sure to include in your index of satisfaction.

- For example, let $S(\alpha)$ be the translation-invariant satisfaction for a portfolio $\alpha$, let $p_T$ be the current asset valuations, and let $w_T$ be the current net asset value. The effective satisfaction should be

$$S(\alpha) = (w_T - p_T' \alpha_{\text{no cash}}) \psi_1 + S(\alpha_{\text{no cash}})$$

where $\psi_1$ is the (certain) satisfaction of one unit of account held in cash.
Rebalancing & Normality

The basic fact asset values cannot be negative\(^1\) means that portfolio managers must deal with an inherent skewness to \(\Psi_{\alpha} = \alpha' M\). This should be either modeled explicitly or handled implicitly by periodically rebalancing the portfolio.

Rebalancing

Say you rebalance your portfolio \(n\) times between \(T\) and \(T + \tau\). Then effectively you replace the skewed random variables \(M = a + BP_{T+\tau}\) for each asset by

\[
M_{\text{bal}} \triangleq \sum_{i=1}^{n} \tilde{M}_i, \quad \tilde{M}_i \sim \tilde{a} + BP_{T+\tau}/n \quad \forall i = 1, \ldots, n
\]

if we can assume i.i.d. inter-period returns.

If \(n\) is sufficiently large, the Central Limit Theorem comes in and \(M_{\text{bal}}\) is approximately normal.

\(^1\)This is termed “limited liability”.

Forecasting GARCH

Conditional heteroskedasticity models give us one-step results such as $m = \mathbb{E}[Y_{T+\tilde{\tau}} | \mathcal{F}_T]$ and $h = \text{var}[Y_{T+\tilde{\tau}} | \mathcal{F}_T]$. But what are we to do if we are interested in a longer investment horizon such as $\tau \triangleq n\tilde{\tau}$?

Scaling Expectation

The conditional expectation scales naturally.

$$
\mathbb{E} \left[ \sum_{i=1}^{n} Y_{T+i\tilde{\tau}} \bigg| \mathcal{F}_T \right] = \sum_{i=1}^{n} \mathbb{E} \left[ Y_{T+i\tilde{\tau}} \bigg| \mathcal{F}_T \right] = nm
$$
Forecasting GARCH

Scaling Variance

Scaling the variance is a bit more involved.

\[
\text{var} \left[ \sum_{i=1}^{n} Y_{T+i\tilde{T}} \mid \mathcal{F}_T \right] = \sum_{i=1}^{n} \text{var} \left[ Y_{T+i\tilde{T}} \mid \mathcal{F}_T \right] = h + \mathbb{E} h_2 + \mathbb{E} h_3 + \cdots + \mathbb{E} h_n
\]

where all of the expectations are conditioned on \( \mathcal{F}_T \). For the asymmetric model we have been working with,

\[
\mathbb{E} h_2 = \omega + h (\beta + \alpha (1 + \gamma^2)) \equiv \omega + h \phi
\]
Scaling Variance

Expected single-period conditional variances further in the future are seen to follow the pattern

\[ E h_i = \omega \left( 1 + \phi + \phi^2 + \cdots + \phi^{i-2} \right) + h\phi^{i-1} \]

Summing these up, we get

\[
\text{var} \left[ \sum_{i=1}^{n} Y_{T+i\tau} \mid \mathcal{F}_T \right] = \frac{1 - \phi^n}{1 - \phi} \left( h - \sigma^2 \right) + n\sigma^2
\]

where \( \sigma^2 \triangleq \frac{\omega}{1 - \phi} \) is the unconditional single-period variance.

- We can see that the GARCH forecast for cumulative variance over \( n \) periods is a natural blend of the short-term and the long-term forecasts.